Principal Bundles, Classifying Spaces, and Obstruction Theory

A <u>fiber bundle</u> with fiber F:

- $\pi : E \to B$ continuous, surjective
- $\forall b \in B$, $\exists$ open nhd U and a homeomorphism

$\varphi : \pi^{-1}(U) \longrightarrow U \times F$ such that
$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & U \times F \arrow[d, "pr_1"] \\ & U \end{tikzcd}$$
commutes.

$\Rightarrow \pi^{-1}(b) \cong F$

(U, φ) local trivialisation.

Could also consider smooth fiber bundles.

<u>Examples:</u>

- Vector bundles, $F = \mathbb{R}^k$
- Covering maps, F discrete
- Hopf fibration, $S^1 \to S^3 \to S^2$

G topological group, a <u>principal G-bundle</u>:

- $\pi : P \to B$ fiber bundle
- Continuous right action $P \times G \to P$ such that:
  1. action preserves π
  2. G acts freely and transitively on each fiber

$\pi^{-1}(b) \cong G$

If G is a Lie group, can also consider smooth principal G-bundles.

<u>Examples:</u> · A regular covering map is a principal G-bundle, G = deck transformations of the cover.

· Hopf fibration $S^1 \to S^3 \to S^2$

$S^1 = U(1)$, $S^3 \subset \mathbb{C}^2$, S^1 acts on the right by scalar mult.

$S^1 \to S^3 \to \mathbb{CP}^1 = S^2$. Generalise : $S^1 \to S^{2n+1} \to \mathbb{CP}^n$,

$S^3 \to \underset{\mathbb{H}^{n+1}}{\overset{}{S^{4n+3}}} \to \mathbb{HP}^n$, $S^3 = Sp(1)$,

$$S^{4n+3} \subset \mathbb{H}^{n+1}$$

· H is a closed Lie subgroup of G, principal H-bundle $H \to G \to G/H$.

There is a natural principal bundle associated to a vector bundle E.

$f: \mathbb{R}^k \to E_b$ $\quad$ $p: E \to B$, $E_b = p^{-1}(b)$

↑ <u>frame</u> if f is an isomorphism. $Fr(E_b) = \{\text{frames for } E_b\}$.

$Fr(E) = \coprod_{b \in B} Fr(E_b)$; $\quad \pi: Fr(E) \to B$.

Local triviality of E $\rightsquigarrow$ natural topology on $Fr(E)$.

Note that $GL(k, \mathbb{R})$ acts on $Fr(E)$: $(b, f) \cdot A = (b, f \circ A)$.

$$\mathbb{R}^k \xrightarrow{A} \mathbb{R}^k \xrightarrow{f} E_b$$

This preserves fibers, and its free and transitive: if f_1, f_2 are frames for E_b, then $f_2 = f_1 \circ A$ for a unique $A \in GL(k,\mathbb{R})$, namely $A = f_1^{-1} \circ f_2$

$$\mathbb{R}^k \xrightarrow{f_2} E_b \xrightarrow{f_1^{-1}} \mathbb{R}^k.$$

$Fr(E) \xrightarrow{\pi} B$ is a principal $GL(k,\mathbb{R})$-bundle. I will instead denote this by $P_{GL(k,\mathbb{R})}(E)$.

One can reconstruct E from $P_{GL(k,\mathbb{R})}(E)$.

$$\left(P_{GL(k,\mathbb{R})}(E) \times \mathbb{R}^k\right)/GL(k,\mathbb{R}), \qquad \begin{aligned}((b,f),v)\cdot A &= ((b,f)\cdot A, A^{-1}_{\;}v) \\ &= ((b, f\circ A), A^{-1}v)\end{aligned}$$

This is a rank k vector bundle, and is isomorphic to E

$$\left(P_{GL(k,\mathbb{R})}(E) \times \mathbb{R}^k\right)/GL(k,\mathbb{R}) \to E$$
$$[((b,f),v)] \longmapsto f(v) \in E_b$$

1–1 correspondence

$$\begin{array}{c} \left\{\begin{array}{c}\text{iso. classes of real} \\ \text{rank } k \text{ vector bundles}\end{array}\right\} \\ \text{associated bundle construction} \uparrow \quad \downarrow \text{frame bundle} \\ \left\{\begin{array}{c}\text{isomorphism classes of} \\ \text{principal } GL(k,\mathbb{R})\text{-bundles}\end{array}\right\}\end{array}$$

<u>Definition:</u> H, G topological groups, $\rho : H \to G$ cont. gp. homomorphism. A <u>reduction of structure group</u> (ROSG) of a principal G-bundle

to H is a principal H-bundle P_H and a map $r : P_H \to P_G$ with $r(ph) = r(p)\rho(h)$.

One can reconstruct P_G from P_H and ρ: $P_G \cong P_H \times_\rho G = (P_H \times G)/H$ where $(p, g) \cdot h = (ph, \rho(h^{-1})g)$.

Note, $\rho : H \to G$ not assumed to be injective. When its not, sometimes we call a RoSG a 'lift' of structure group.

When $\rho : G \to GL(n, \mathbb{R})$ and $P_{GL(n,\mathbb{R})} = P_{GL(n,\mathbb{R})}(TM)$, a RoSG to G is called a <u>G-structure</u> on M.

<u>Examples:</u> $E \to B$ rank k v.b.

- E equipped with an orientation.

$P_{GL^+(k,\mathbb{R})}(E) = \{ (b, f) \in P_{GL(k,\mathbb{R})}(E) \mid f : \mathbb{R}^k \to E_b,\ f \text{ preserves orientation} \}$

$A \in GL(k, \mathbb{R})$, f preserves orientation, $f \circ A$ preserves orientation $\Leftrightarrow$ A preserves orientation $\Leftrightarrow$ $\det(A) > 0$ $\Leftrightarrow$ $A \in GL^+(k, \mathbb{R})$

$P_{GL^+(k,\mathbb{R})}(E)$ is a principal $GL^+(k, \mathbb{R})$-bundle, and $P_{GL^+(k,\mathbb{R})}(E) \hookrightarrow P_{GL(k,\mathbb{R})}(E)$ is a RoSG of $P_{GL(k,\mathbb{R})}(E)$ to $GL^+(k, \mathbb{R})$.

Conversely, if $P_{GL^+(k,\mathbb{R})} \xrightarrow{r} P_{GL(k,\mathbb{R})}(E)$ is a RoSG, define an orientation on E as follows: $(b, f) \in r(P_{GL^+(k,\mathbb{R})})$ equip E_b with the orientation $f(e_1) \wedge \dots \wedge f(e_k) \in \Lambda^k E_b$.

Given any other $(b, f') \in r(P_{GL^+(k,\mathbb{R})})$, $f'(e_1) \wedge \dots \wedge f'(e_k)$ defines the same orientation.

$P_{GL(k,\mathbb{R})}(E)$ admits a RoSG to $GL^+(k,\mathbb{R})$ $\Leftrightarrow$ E admits an orientation.

M admits a $GL^+(n,\mathbb{R})$-structure $\Leftrightarrow$ M is orientable.

- E bundle metric

$P_{O(k)}(E) = \{ (b, f) \in P_{GL(k,\mathbb{R})}(E) \mid f : \mathbb{R}^k \to E_b,\ f \text{ isometry} \}$

is a principal $O(k)$-bundle, $P_{O(k)}(E) \hookrightarrow P_{GL(k,\mathbb{R})}(E)$ is a RoSG of $P_{GL(k,\mathbb{R})}(E)$ to $O(k)$.

Conversely, a RoSG $P_{O(k)} \xrightarrow{r} P_{GL(k,\mathbb{R})}(E)$ defines a bundle metric on E:

$$g_b(u, v) = g_{Eucl}(f^{-1}(u), f^{-1}(v)) \qquad (b,f) \in r(P_{O(k)})$$

M admits an $O(n)$-structure $\Leftrightarrow$ M admits a Riemannian metric.

- M admits an $SO(n)$-structure $\Leftrightarrow$ M admits a Riemannian metric and M is orientable

- If $k = 2m$, $P_{GL(2m,\mathbb{R})}(E)$ admits a RoSG to $GL(m,\mathbb{C})$ $\Leftrightarrow$ E admits an almost complex structure ($J : E \to E$, $J \circ J = -id_E$).

- $P_{SO(2m)}(E)$ admits a RoSG to $U(m)$ $\Leftrightarrow$ E admits an acs compatible with the bundle metric and inducing the given orientation.

- $GL(k-\ell, \mathbb{R}) \subset GL(k, \mathbb{R}) \iff E \cong E_0 \oplus \varepsilon^\ell$ ← trivial

- $GL(k-\ell, \mathbb{R}) \times GL(\ell, \mathbb{R}) \subset GL(k, \mathbb{R}) \iff E = E_0 \oplus E_1$ $rk(E_0) = k-\ell$, $rk(E_1) = \ell$

- $G_2 \subset SO(7) \iff G_2$-structure

- $Spin(7) \subset SO(8) \iff Spin(7)$-structure

- $Spin(n) \xrightarrow{f} SO(n)$ double covering $\iff E$ has a spin structure

If $G < GL(k, \mathbb{R})$, $P_{GL(k,\mathbb{R})}(E)$ admits a ROSG to G $\iff$ there is a collection of local trivialisations whose transition functions take values in G.

When does a principal G-bundle admit a ROSG to H?

<u>Theorem:</u> $H < G$. If $P_G \to B$ admits a ROSG to H, then $P_G / H \longrightarrow B$ (a fiber bundle with fiber G/H) has a section.

Conversely, if H is admissible*, then sections of $P_G/H \to B$ correspond to ROSG of P_G to H.

$H < G$ admissible if $H \to G \to G/H$ principal H-bundle (true for H closed Lie subgroup of G)

<u>Examples:</u> · $P_{GL(n,\mathbb{R})}(TM) / GL^+(n,\mathbb{R}) \longrightarrow M$

has fiber $GL(n,\mathbb{R}) / GL^+(n,\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$

This is the orientable double cover of M.

· $P_{SO(2n)}(TM) / U(n) \longrightarrow M$ twistor space of (M, g), oriented.

Sections $\longleftrightarrow$ acs on M compatible with g inducing the given orientation,