Principal Bundles, Classifying Spaces, and Obstruction Theory

Last time:
- Principal bundles
- Reduction of structure group
- $H < G$ admissible. $P_G \to B$ admits ROSG to H $\Leftrightarrow$ $P_G / H \to B$ has a section.

How can we tell if $P_G / H \to B$ has a section?

Obstruction Theory

Definition: $p : E \to B$ cont., space Y

p has the homotopy lifting property (HLP) with respect to Y if for any homotopy $H : Y \times [0,1] \to B$ and a lift $\tilde{h}_0 : Y \to E$ of $h_0 : Y \to B$, $h_0(y) = H(y,0)$, i.e. $p \circ \tilde{h}_0 = h_0$, there is a homotopy $\tilde{H} : Y \times [0,1] \to E$ such that $p \circ \tilde{H} = H$ and $\tilde{h}_0 = \tilde{H}(\cdot, 0)$.

$i_0(y) = (y, 0)$

$$\begin{tikzcd}
Y \arrow[r, "\tilde{h}_0"] \arrow[d, "i_0"'] \arrow[dr, "h_0"] & E \arrow[d, "p"] \\
Y \times [0,1] \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & B
\end{tikzcd}$$

$\tilde{H}$ not necessarily unique

Example: HLP for $Y = \{y_0\}$ corresponds to the path lifting property. So covering maps satisfy HLP for singletons.

Definition: A surj. cont. map $p : E \to B$ is called a (Serre) fibration if p has the HLP for all CW complexes Y.

<u>Exercise:</u> Show $pr_1 : B \times F \to B$ has the HLP for any space, and hence is a fibration.

<u>Proposition:</u> Fiber bundles are fibrations.

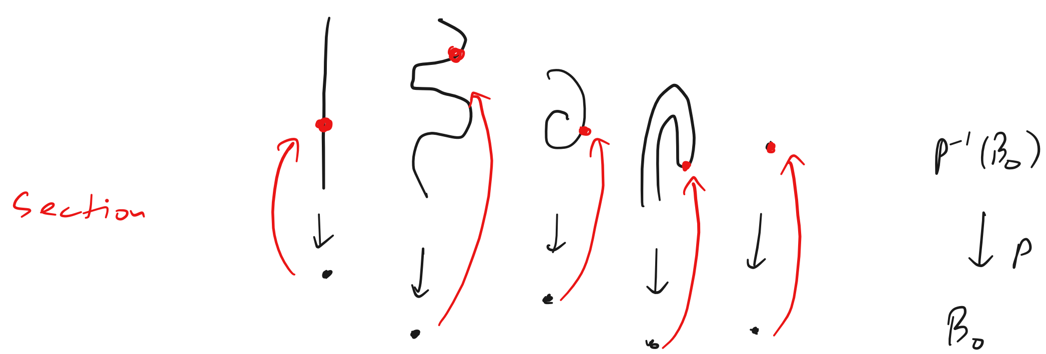
Fibrations are the homotopy theoretic analogue of bundles.
In particular, fibers are all homotopy equivalent.

<u>Theorem:</u> $p: E \to B$ fibration, $b_0 \in B$, $x_0 \in F := p^{-1}(b_0)$. There is a l.e.s.

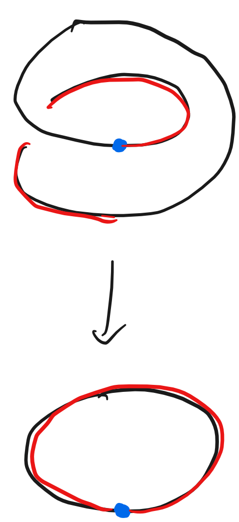
$$\cdots \to \pi_{n+1}(B, b_0) \xrightarrow{\partial} \pi_n(F, x_0) \xrightarrow{i_*} \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, x_0) \to \cdots$$

$i : F \hookrightarrow E$.

If $p: E \to B$ a fibration, B CW complex. Over B_0, the 0-skeleton of B, p has a section (choose a point in each fiber).



Obstruction theory studies the obstructions to extending a section built inductively on the cells of B.



Doesn't extend to a section over S^1

Suppose σ is a section of $p: E \to B$ defined over the n-skeleton B_n.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ B_n \arrow[ur, "\sigma"] \arrow[r, hook, "i"'] & B \end{tikzcd}$$

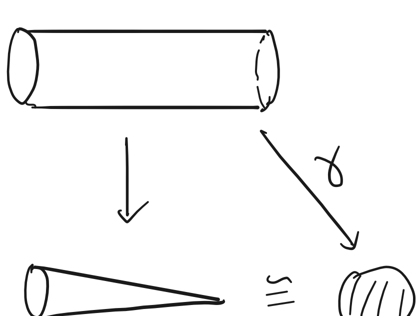
inclusion

Choose an $(n+1)$-cell e_α^{n+1} in B with attaching map $g := g_\alpha^{n+1} : \partial e_\alpha^{n+1} = S^n \longrightarrow B_n$.

$$\begin{tikzcd} & & E \arrow[d, "p"] \\ S^n \arrow[urr, bend left, "\tilde{\sigma}"] \arrow[r, "g"'] & B_n \arrow[ur, "\sigma"] \arrow[r, hook, "i"'] & B \end{tikzcd}$$

$\tilde{\sigma} = \sigma \circ g$

$i \circ g : S^n \to B$ extends to D^{n+1} (namely e_α^{n+1}), so there is a map $h: D^{n+1} \to B$ with $h|_{S^n} = i \circ g$. This gives a homotopy from $i \circ g$ and a constant map: $H : S^n \times [0,1] \to B$, $H(p,t) = h((1-t)p)$. So H is a homotopy from $h_0 = H(\cdot, 0) = h|_{S^n} = i \circ p$ and h_1 which is the constant map with value $c := h(0)$.



$$\begin{tikzcd} S^n \times [0,1] \arrow[d, "\gamma"'] \arrow[dr, "H"] & \\ D^{n+1} \arrow[r, "h"'] & B \end{tikzcd}$$

$\gamma(p,t) = (1-t)p$

$p \circ \tilde{\sigma} = i \circ g = h_0$ so $\tilde{\sigma}$ is a lift of h_0. By the HLP, there is a homotopy $\tilde{H} : S^n \times [0,1] \to E$ such that $\tilde{h}_0 = \tilde{\sigma}$ and $p \circ \tilde{H} = H$.

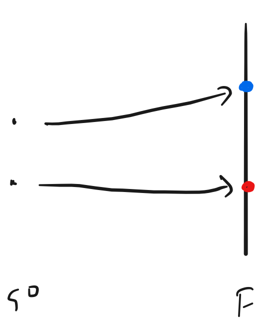
$$\begin{tikzcd}
S^n \times [0,1] \arrow[r, "\tilde{H}"] \arrow[d, "\gamma"'] \arrow[dr, "H"] & E \arrow[d, "p"] \\
D^{n+1} \arrow[r, "h"'] & B
\end{tikzcd}$$

$\tilde{h}_1 = \tilde{H}(\cdot, 1)$ satisfies $p \circ \tilde{h}_1 = h_1 \equiv c$, so $\tilde{h}_1 : S^n \to F := p^{-1}(c)$.

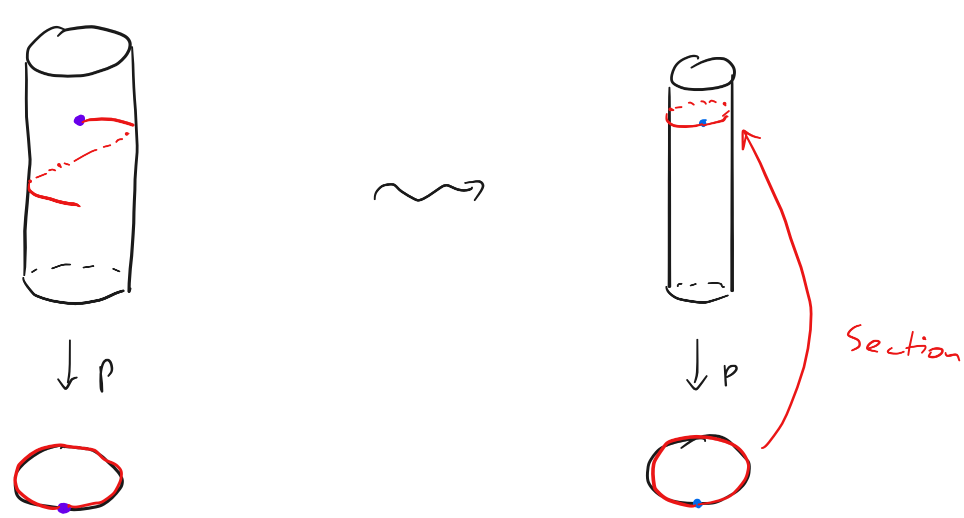
If $\tilde{h}_1$ were constant, then $\tilde{H} : S^n \times [0,1] \to E$ descends to a map $\sigma_\alpha : D^{n+1} \to E$ which satisfies $p \circ \sigma_\alpha = h$.

$$\begin{tikzcd}
S^n \times [0,1] \arrow[r, "\tilde{H}"] \arrow[d, "\gamma"'] \arrow[dr, "H"] & E \arrow[d, "p"] \\
D^{n+1} \arrow[r, "h"'] \arrow[ur, dashed, "\sigma_\alpha"] & B
\end{tikzcd}$$

That is, if $\tilde{h}_1$ is constant, σ_α is a section of p over e_α^{n+1} extending $\sigma|_{\partial e_\alpha^{n+1}}$. If $\tilde{h}_1 : S^n \to F$ is homotopic to a constant, we can modify $h, H, \tilde{H}$ so that $\tilde{h}_1$ is constant.



Not constant, but homotopic to a constant



The lift $\tilde{H}$ is not unique, so neither is $\tilde{h}_1$. If $\tilde{H}'$ is another lifted homotopy, we get a map $\tilde{h}_1' : S^n \to F$. Note that $\tilde{h}_1$ and $\tilde{h}_1'$ are homotopic as they are both homotopic to $\tilde{h}_0 = \tilde{\sigma} = \tilde{h}_0'$.

So to each $(n+1)$-cell we can associate $[\tilde{h}_1] \in [S^n, F]$ which is trivial $\Leftrightarrow$ σ extends over the $(n+1)$-cell. In general $[S^n, F] \neq \pi_n(F)$ (basepoints). We say F is <u>n-simple</u> if $[S^n, F] = \pi_n(F)$. Examples include F simply connected, and G/H for H a closed Lie subgroup of G.

One further complication: for different cells, we get elements in homotopy groups of different fibers. We can't compare elements of $\pi_n(F)$ and $\pi_n(F')$. This can be resolved using a local coefficient system (we will ignore this subtlety as it won't apply in our examples).

So for any $(n+1)$-cell e^{n+1}_α we associate $[\tilde{h}_1] \in \pi_n(F)$. This is a (cellular) cochain with values in $\pi_n(F)$. Miraculously, it is a cocycle. So we get an element $\gamma_{n+1} \in H^{n+1}(B; \pi_n(F))$.

<u>Theorem:</u> $p : E \to B$ fibration, B CW complex, σ a section over B_n. If F is n-simple, then $\gamma_{n+1} \in H^{n+1}(B; \pi_n(F))$ is defined. If $\gamma_{n+1} = 0$, then σ can be redefined over the n-skeleton relative to the $(n-1)$-skeleton, such that it extends to a section over the $(n+1)$-skeleton of B.