Principal Bundles, Classifying Spaces, and Obstruction Theory

Last time: $p: E \to B$ fibration, B CW complex, σ section over B_n.

$\gamma_{n+1} \in H^{n+1}(B; \pi_n(F))$ vanishes $\Leftrightarrow$ there is a section over B_{n+1}.

If $\gamma_{n+1} = 0$, γ_{n+2} is defined. If all obstructions vanish, there is a section of $p: E \to B$.

If $\pi_n(F) = 0$ $n \leq k-1$, and $\pi_k(F) \neq 0$, we call $\gamma_{k+1} \in H^{k+1}(B; \pi_k(F))$ the <u>primary obstruction</u>.

<u>Examples:</u>

- $p: E \to B$ a vector bundle, $F = \mathbb{R}^d$, so $\pi_k(\mathbb{R}^d) = 0$. All obstructions to the existence of a section vanish. So every vector bundle has a section (knew that already, e.g. zero section).

- More generally, if $p: E \to B$ is a fibration with contractible fiber F, then p has a section. For instance, for a vector bundle $V \to B$, $P_{GL(d,\mathbb{R})}(V) \to B$ has a ROSG to $O(d)$ $\Leftrightarrow$ $P_{GL(d,\mathbb{R})}(V)/O(d) \to B$ has a section $\Leftrightarrow$ V admits a bundle metric. The fiber of $P_{GL(d,\mathbb{R})}(V)/O(d) \to B$ is $\dfrac{GL(d,\mathbb{R})}{O(d)}$ which is contractible since $O(d)$ is a maximal compact subgroup of $GL(d,\mathbb{R})$. So every $V \to B$ admits a bundle metric.

- $P_{GL(d,\mathbb{R})}(V) \to B$ admits a ROSG to $GL^+(d,\mathbb{R})$
  $\Leftrightarrow$ $P_{GL(d,\mathbb{R})}(V)/GL^+(d,\mathbb{R}) \to B$ has a section
  $\Leftrightarrow$ V is orientable.

The fiber of $P_{GL(d,\mathbb{R})}(V)/GL^+(d,\mathbb{R}) \to B$ is $\dfrac{GL(d,\mathbb{R})}{GL^+(d,\mathbb{R})} \cong \mathbb{Z}_2$.

$\pi_0(\mathbb{Z}_2) = \mathbb{Z}_2$, $\pi_n(\mathbb{Z}_2) = 0$ $n > 0$. There is one obstruction in $H^1(B; \pi_0(F)) = H^1(B; \mathbb{Z}_2)$. This is called the <u>first Stiefel-Whitney class</u> of V, denoted $w_1(V)$. So V is orientable $\Leftrightarrow$ $w_1(V) = 0$.

- $P_{SO(d)}(V) \to B$ admits a ROSG to $SO(d-1)$

$\Leftrightarrow$ $P_{SO(d)}(V)/SO(d-1) \to B$ has a section

$\Leftrightarrow$ V has a nowhere-zero section

$\Leftrightarrow$ $V \cong V_0 \oplus \varepsilon^1$

The fiber $P_{SO(d)}(V)/SO(d-1) \to B$ is $\frac{SO(d)}{SO(d-1)} = S^{d-1}$. The primary obstruction belongs to $H^d(B; \pi_{d-1}(S^{d-1})) = H^d(B; \mathbb{Z})$. This is called the <u>Euler class</u> of V, denoted $e(V)$.

- M^4 oriented admits an almost complex structure

$\Leftrightarrow$ $P_{SO(4)}(TM) \to M$ admits a ROSG to $U(2)$

$\Leftrightarrow$ $P_{SO(4)}(TM)/U(2) \to M$ has a section.

The fiber is $\frac{SO(4)}{U(2)} = S^2$. So there are two obstructions to the existence of an ACS on M^4: $\gamma_3 \in H^3(M; \pi_2(S^2)) = H^3(M; \mathbb{Z})$ and $\gamma_4 \in H^4(M; \pi_3(S^2)) = H^4(M; \mathbb{Z})$. What are they?

How do we identify obstructions?

<u>Classifying Spaces</u>

<u>Motivating Example:</u> M^n smooth manifold, $TM \to M$ rank n

Whitney $\Rightarrow$ embedding $f: M \to \mathbb{R}^N$, so $f_*: TM \to T\mathbb{R}^N = \mathbb{R}^N \times \mathbb{R}^N$.

We obtain a map $\phi: M \to Gr_n(\mathbb{R}^N)$ (Gauss map).

$$p \mapsto pr_2(f_*(T_pM))$$

The tautological bundle $V_n \xrightarrow{\pi} Gr_n(\mathbb{R}^N)$,

$V_n = \{(W, v) \in Gr_n(\mathbb{R}^N) \times \mathbb{R}^N \mid v \in W\}$, $\pi(W, v) = W$, $\pi^{-1}(W) = W$.

ϕ has the property that $\phi^* V_n \cong TM$. Every tangent bundle is a pullback of the tautological bundle. In fact, for N large enough, every bundle over M is the pullback of V_n by some map $\phi: M \to \mathbb{R}^N$.

Returning to principal G-bundles, is there an analogue?

<u>Lemma:</u> $f_1, f_2: X \to Y$, $P_G \to Y$ principal G-bundle

If f_1 and f_2 are homotopic, then $f_1^* P_G$ and $f_2^* P_G$ are isomorphic.

<u>Definition:</u> A principal G-bundle $P_G \to B$ is called a <u>universal G-bundle</u> if the map $[X, B] \to Prin_G(X) = \{\text{iso. classes of prin. } G\text{-bundles}\}$

$$[f] \mapsto [f^* P_G]$$

is a bijection. Given a principal G-bundle $Q \to X$, a map $\phi: X \to B$ with $\phi^* P_G = B$ is called a <u>classifying map</u>. We call B a <u>classifying space</u> for G.

<u>Theorem (Milnor):</u> G top. group, $\exists$ a universal G-bundle.

The classifying space is unique up to homotopy equivalence, and the universal G-bundle is unique up to isomorphism.

We denote the homotopy type of the classifying space by BG.

<u>Proposition:</u> $P_G \to B$ universal $\iff$ P_G is weakly contractible (i.e. $\pi_k(P_G) = 0$ $\forall k \geq 0$)

Long exact sequence in homotopy groups $\Rightarrow$ $\pi_k(BG) = \pi_{k-1}(G)$

<u>Examples:</u> • $P_{GL(n,\mathbb{R})}(V_n) \to Gr_n(\mathbb{R}^N)$ is the Stiefel manifold $GL(N,\mathbb{R})/GL(N-n)$ which is $(N-n-1)$-connected.

It follows that $P_{GL(n,\mathbb{R})}(V_n) \to Gr_n(\mathbb{R}^\infty)$ is a universal $GL(n,\mathbb{R})$-bundle.

Equip V_n with a bundle metric, $P_{O(n)}(V_n) \to Gr_n(\mathbb{R}^\infty)$ is a universal $O(n)$-bundle.

$Gr_n(\mathbb{R}^\infty) = B\,GL(n,\mathbb{R}) = B\,O(n)$.

In general, $\rho: H \to G$ cont. group homomorphism, $B\rho: BH \to BG$ and if ρ is a hom. eq., so is $B\rho$.

• X aspherical (cont. universal cover), $\Gamma = \pi_1(X)$, $\Gamma \to \tilde{X} \to X$ is a universal Γ-bundle since $\tilde{X}$ is contractible.

e.g. $\mathbb{Z} \to \mathbb{R} \to S^1$ is the universal $\mathbb{Z}$-bundle.

Given a principal G-bundle $Q \to X$, $\phi: X \to BG$ classifying map. and $\rho: H \to G$.

Q admits a RoSG to H $\iff$ $\phi: X \to BG$ lifts through $B\rho: BH \to BG$ to a map $\tilde{\phi}: X \to BH$.

$$\begin{tikzcd} & BH \arrow[d, "B\rho"] \\ X \arrow[ur, dashed, "\tilde{\phi}"] \arrow[r, "\phi"'] & BG \end{tikzcd}$$

If $\alpha \in H^k(BG;\mathbb{Z})$ and $(B\rho)^*\alpha = 0$, then if ϕ has a lift $\tilde{\phi}$, we have $\phi^*\alpha = (B\rho \circ \tilde{\phi})^*\alpha = \tilde{\phi}^*(B\rho)^*\alpha = 0$. So $\phi^*\alpha = 0$ is a necessary condition for the existence of a lift $\tilde{\phi}$ and hence a RoSG to H. This is related to the obstructions we mentioned previously.

If $H < G$, then $BH \to BG$ is a fiber bundle with fiber G/H, and $\phi^* BH \to X$ is isomorphic to $Q/H \to X$.

$Q \to X$ principal G-bundle with classifying map ϕ

$$\begin{tikzcd} \phi^* BH \arrow[r] \arrow[d] & BH \arrow[d, "B\rho"] \\ X \arrow[u, bend left, "\sigma"] \arrow[ur, "\tilde{\phi}"] \arrow[r, "\phi"'] & BG \end{tikzcd}$$

A lift $\tilde{\phi}$ exists $\Leftrightarrow$ a section σ exists.