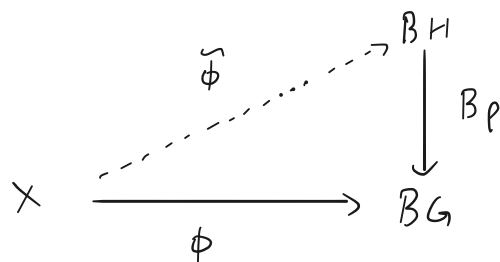


Principal Bundles, Classifying Spaces, and Obstruction Theory

Last time:

- How do we identify obstructions?
- Classifying Spaces
- $Q \rightarrow X$ a principal G -bundle, $\rho: H \rightarrow G$
 Q admits a lift to H \Leftrightarrow its classifying map $\phi: X \rightarrow BG$
 lifts to a map $\tilde{\phi}: X \rightarrow BH$.
- If $\tilde{\phi}$ exists, and $\alpha \in H^k(BG; \mathbb{R})$ with $(B\rho)^*\alpha = 0$,
 then $\phi^*\alpha = 0$.



Definition: $\alpha \in H^k(BG; \mathbb{R})$ is called a universal characteristic class.

$Q \rightarrow X$ is a principal G -bundle with classifying map $\phi: X \rightarrow BG$,
 we set $\alpha(Q) = \phi^*\alpha \in H^k(X; \mathbb{R})$.

If $(B\rho)^*\alpha = 0$, then $\alpha(Q) = \phi^*\alpha = (B\rho \circ \tilde{\phi})^*\alpha = \tilde{\phi}^*(B\rho)^*\alpha = 0$.

So $\alpha(Q) = 0$ is a necessary condition for the existence of a lift.

Example: $\rho: GL^+(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$ inclusion

$B\rho: BGL^+(n, \mathbb{R}) \rightarrow BGL(n, \mathbb{R})$ is a fiber bundle with fiber

$GL(n, \mathbb{R}) / GL^+(n, \mathbb{R}) \cong \mathbb{Z}_2$. Note that $\pi_1(BGL(n, \mathbb{R})) = \pi_0(GL(n, \mathbb{R})) = \mathbb{Z}_2$

so $H^1(BGL(n, \mathbb{R}); \mathbb{Z}_2) = \text{Hom}(\pi_1(BGL(n, \mathbb{R})), \mathbb{Z}_2) = \mathbb{Z}_2$. On the other hand,

$\pi_1(BGL^+(n, \mathbb{R})) = \pi_0(GL^+(n, \mathbb{R})) = 0$ so $H^1(BGL^+(n, \mathbb{R}); \mathbb{Z}_2) = 0$.

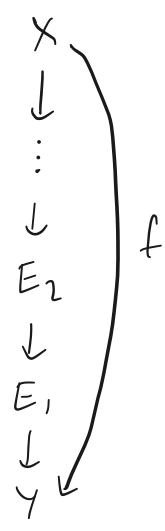
If $\alpha \in H^1(BGL(n, \mathbb{R}); \mathbb{Z}_2)$ is non-zero, then $(B\rho)^*\alpha = 0$, so $\alpha(Q) = 0$ is a necessary condition for Q admitting a $\mathbb{R}OSG$ to $GL^+(n, \mathbb{R})$.

In fact, $\alpha = w_1$ the universal first Stiefel-Whitney class, so $\alpha(Q) = w_1(Q)$ which is what we found using obstruction theory.

O.T. tells us that there is a $\mathbb{R}OSG \iff w_1(Q) = 0$. However, there are many classes $\alpha \in H^k(BGL(n, \mathbb{R}); \mathbb{Z}_2)$ with $(B\rho)^*\alpha = 0$, e.g. $\alpha = w_1^k$.

In general, how do we find a 'minimal set' of universal characteristic classes $\alpha \in H^k(BG; \mathbb{Z})$ with $(B\rho)^*\alpha = 0$ which determine exactly when there is a lift?

Definition: $f: X \rightarrow Y$ continuous. A Moore-Postnikov tower for f is a sequence of maps



where $X \rightarrow E_n$ induces an isomorphism of π_i for $i < n-1$ and is surjective on π_{n-1} , and $E_n \rightarrow Y$ induces an isomorphism of π_i for $i > n$ and is injective on π_{n-1} .

$\pi_0(E_n)$	$\pi_1(E_n)$...	$\pi_{n-2}(E_n)$	$\pi_{n-1}(E_n)$	$\pi_n(E_n)$	$\pi_{n+1}(E_n)$...
\parallel	\parallel		\parallel		\parallel	\parallel	
$\pi_0(X)$	$\pi_1(X)$		$\pi_{n-2}(E_n)$		$\pi_n(Y)$	$\pi_{n+1}(Y)$	

These towers exist for any f .

It follows from the definition that $E_n \rightarrow E_{n-1}$ induces:

- iso. on π_i $i \neq n-2, n-1$
- inj. on π_{n-2}
- surj. on π_{n-1} .

We can choose $E_n \rightarrow E_{n-1}$ to be a fibration. If F is the fiber, then it follows from the long exact sequence in homotopy groups that $\pi_i(F) = 0$ for $i \neq n-2$.

Definition: Let Z be a top. space with $\pi_i(Z) = 0$ for $i \neq m$ and $\pi_m(Z) \cong G$.

We call Z an Eilenberg - MacLane space of type $K(G, m)$.

Such spaces are unique up to homotopy equivalence.

F , the fiber of $E_n \rightarrow E_{n-1}$, is a $K(G, n-2)$ for some G .

Example: $f: \{*\} \rightarrow S^2$. The Moore - Postnikov tower is

$$\begin{array}{c} \{*\} \\ \downarrow \\ \vdots \\ \downarrow \\ E_3 \\ \downarrow \\ E_2 \\ \downarrow \\ E_1 \\ \downarrow \\ S^2 \end{array}$$

$$\pi_i(E_n) \cong \pi_i(\{*\}) \quad i \leq n-2, \quad \pi_i(E_n) \cong \pi_i(S^2) \quad i \geq n$$

$$\pi_0(\{*\}) = \pi_0(S^2) = 0 \quad \rightsquigarrow \quad E_1 = S^2$$

$$\pi_1(\{*\}) = \pi_1(S^2) = 0 \quad \rightsquigarrow \quad E_2 = S^2$$

$$\pi_2(\{*\}) = 0, \quad \pi_2(S^2) \cong \mathbb{Z} \neq 0 \quad E_3 \text{ cannot be } S^2$$

$$\pi_i(\{*\}) \rightarrow \pi_i(E_3) \text{ is surjective} \quad i = n-1 = 3-1 = 2$$

$$\pi_2(S^2) = 0 \Rightarrow \pi_2(E_3) = 0$$

$$F \rightarrow E_3$$

$$\downarrow$$

$$S^2 = E_2$$

F is a $K(G, 1)$

$$G \cong \mathbb{Z}$$

F is a $K(\mathbb{Z}, 1) \rightsquigarrow F = S^1$

$$S^1 \rightarrow S^3 = E_3$$

$$\downarrow$$

$$S^2$$

Hopf fibration.

In this example, $E_3 \rightarrow E_2$ is a principal bundle.

In general, if the local coefficient system is trivial, then the maps $E_n \rightarrow E_{n-1}$ are all 'principal fibrations'. Just as with principal bundles, principal fibrations have classifying spaces.

$$K(G, n-2) \rightarrow E_n$$

$$\downarrow$$

$$E_{n-1}$$

is classified by a map $E_{n-1} \rightarrow BK(G, n-2)$.

Recall $\pi_n(BG) \cong \pi_{n-1}(G)$. It follows that $BK(G, n-2) = K(G, n-1)$.

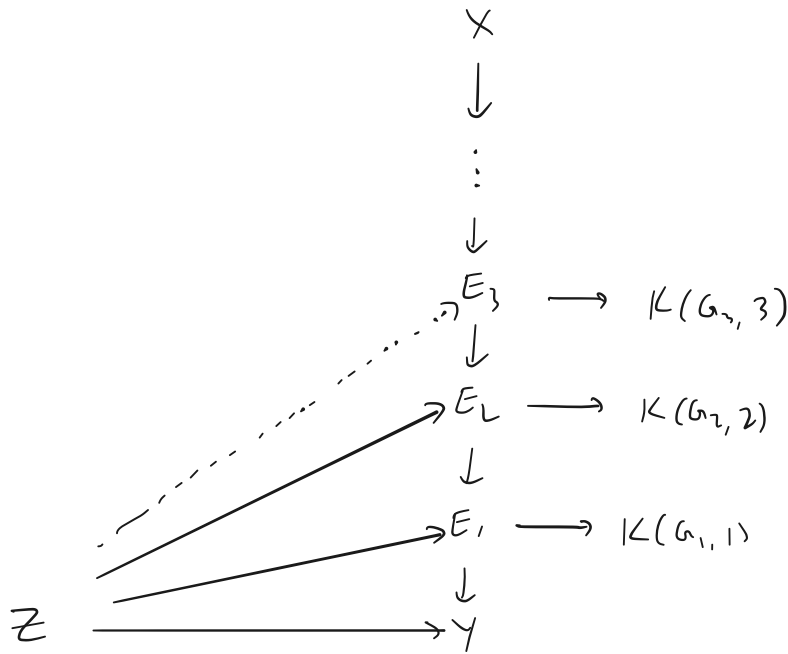
$E_{n-1} \rightarrow K(G, n-1)$ classifying $E_n \rightarrow E_{n-1}$.

$$[X, K(G, m)] \rightarrow H^m(X; G)$$

$$i_m \in H^m(K(G, m); G)$$

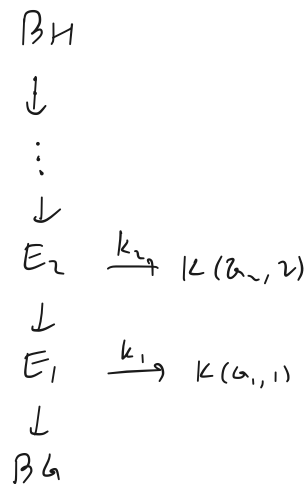
$$[f] \longmapsto f^* i_m$$

Natural transformation.



$Z \rightarrow E_2$ lifts to $Z \rightarrow E_3$ if and only if the composition $Z \rightarrow E_2 \rightarrow K(G_n, 2)$ is nullhomotopic, i.e. the corresponding cohomology class in $H^2(Z; G_n)$ is zero.

We can study the Moore-Postnikov tower for $Bp: BH \rightarrow BG$



$k_n: E_n \rightarrow K(G_n, n)$ corresponds to a class $k_n \in H^n(E_n; G_n)$ which vanishes when pulled back to BH . Sometimes called 'Postnikov invariants of f '.

These are exactly the 'minimal set' of universal classes we desired. The pullback of these classes are the obstruction classes.