Principal Bundles, Classifying Spaces, and Obstruction Theory

Last time:
- How do we identify obstructions?
- Classifying Spaces
- $Q \to X$ a principal G-bundle, $\rho : H \to G$

  Q admits a RoSG to H $\iff$ its classifying map $\phi : X \to BG$ lifts to a map $\tilde{\phi} : X \to BH$.
- If $\tilde{\phi}$ exists, and $\alpha \in H^k(BG; R)$ with $(B\rho)^* \alpha = 0$, then $\phi^* \alpha = 0$.

$$\begin{tikzcd} & BH \arrow[d, "B\rho"] \\ X \arrow[ur, dashed, "\tilde{\phi}"] \arrow[r, "\phi"'] & BG \end{tikzcd}$$

<u>Definition:</u> $\alpha \in H^k(BG; R)$ is called a <u>universal characteristic class</u>. $Q \to X$ is a principal G-bundle with classifying map $\phi : X \to BG$, we set $\alpha(Q) = \phi^* \alpha \in H^k(X; R)$.

If $(B\rho)^* \alpha = 0$, then $\alpha(Q) = \phi^* \alpha = (B\rho \circ \tilde{\phi})^* \alpha = \tilde{\phi}^* (B\rho)^* \alpha = 0$.

So $\alpha(Q) = 0$ is a necessary condition for the existence of a lift.

<u>Example:</u> $\rho : GL^+(n, \mathbb{R}) \to GL(n, \mathbb{R})$ inclusion

$B\rho : BGL^+(n, \mathbb{R}) \to BGL(n, \mathbb{R})$ is a fiber bundle with fiber $GL(n, \mathbb{R}) / GL^+(n, \mathbb{R}) \cong \mathbb{Z}_2$. Note that $\pi_1(BGL(n, \mathbb{R})) = \pi_0(GL(n, \mathbb{R})) = \mathbb{Z}_2$

So $H^1(BGL(n, \mathbb{R}); \mathbb{Z}_2) = \mathrm{Hom}(\pi_1(BGL(n, \mathbb{R})), \mathbb{Z}_2) = \mathbb{Z}_2$. On the other hand, $\pi_1(BGL^+(n, \mathbb{R})) = \pi_0(GL^+(n, \mathbb{R})) = 0$ so $H^1(BGL(n, \mathbb{R}); \mathbb{Z}_2) = 0$.

If $\alpha \in H^1(BGL(n,\mathbb{R});\mathbb{Z}_2)$ is non-zero, then $(B\rho)^*\alpha = 0$, so $\alpha(Q) = 0$ is a necessary condition for Q admitting a ROSG to $GL^+(n,\mathbb{R})$.

In fact, $\alpha = w_1$ the universal first Stiefel-Whitney class, so $\alpha(Q) = w_1(Q)$ which is what we found using obstruction theory.

O.T. tells us that there is a ROSG $\iff w_1(Q) = 0$. However, there are many classes $\alpha \in H^k(BGL(n,\mathbb{R});R)$ with $(B\rho)^*\alpha = 0$, e.g. $\alpha = w_1{}^k$.

In general, how do we find a 'minimal set' of universal characteristic classes $\alpha \in H^k(BG;R)$ with $(B\rho)^*\alpha = 0$ which determine exactly when there is a lift?

<u>Definition:</u> $f: X \to Y$ continuous. A <u>Moore-Postnikov tower</u> for f is a sequence of maps

$$\begin{tikzcd} X \arrow[d] \arrow[dddddd, bend left, "f"] \\ \vdots \arrow[d] \\ E_2 \arrow[d] \\ E_1 \arrow[d] \\ Y \end{tikzcd}$$

where $X \to E_n$ induces an isomorphism of π_i for $i < n-1$ and is surjective on π_{n-1}, and $E_n \to Y$ induces an isomorphism of π_i for $i \geq n$ and is injective on π_{n-1}.

$$\begin{array}{ccccccc} \pi_0(E_n) & \pi_1(E_n) & \cdots & \pi_{n-2}(E_n) & \pi_{n-1}(E_n) & \pi_n(E_n) & \pi_{n+1}(E_n) \quad \cdots \\ \cong & \cong & & \cong & & \cong & \cong \\ \pi_0(X) & \pi_1(X) & & \pi_{n-2}(E_n) & & \pi_n(Y) & \pi_{n+1}(Y) \end{array}$$

These towers exist for any f.

It follows from the definition that $E_n \to E_{n-1}$ induces :
- iso. on π_i $i \neq n-2, n-1$
- inj. on π_{n-2}
- surj. on π_{n-1}.

We can choose $E_n \to E_{n-1}$ to be a fibration. If F is the fiber, then it follows from the long exact sequence in homotopy groups that $\pi_i(F) = 0$ for $i \neq n-2$.

<u>Definition:</u> Let Z be a top. space with $\pi_i(Z) = 0$ for $i \neq m$ and $\pi_m(Z) \cong G$. We call Z an <u>Eilenberg - MacLane space</u> of type $K(G,m)$.

Such spaces are unique up to homotopy equivalence.

F, the fiber of $E_n \to E_{n-1}$, is a $K(G, n-2)$ for some G.

<u>Example:</u> $f : \{*\} \to S^2$. The Moore - Postnikov tower is

$$\begin{array}{c} \{*\} \\ \downarrow \\ \vdots \\ \downarrow \\ E_3 \\ \downarrow \\ E_2 \\ \downarrow \\ E_1 \\ \downarrow \\ S^2 \end{array}$$

$\pi_i(E_n) \cong \pi_i(\{*\})$ $i \leq n-2$, $\pi_i(E_n) \cong \pi_i(S^2)$ $i \geq n$

$\pi_0(\{*\}) = \pi_0(S^2) = 0 \rightsquigarrow E_1 = S^2$

$\pi_1(\{*\}) = \pi_1(S^2) = 0 \quad \rightsquigarrow \quad E_2 = S^2$

$\pi_2(\{*\}) = 0, \quad \pi_2(S^2) \cong \mathbb{Z} \neq 0 \qquad E_3$ cannot be S^2

$\pi_i(\{*\}) \to \pi_i(E_3)$ is surjective $\quad i = n-1 = 3-1 = 2$

$\pi_2(\{*\}) = 0 \Rightarrow \pi_2(E_3) = 0$

$$\begin{array}{ccc} F & \to & E_3 \\ & & \downarrow \\ & & S^2 = E_2 \end{array}$$

F is a $K(G,1)$

$G \cong \mathbb{Z}$

F is a $K(\mathbb{Z},1) \rightsquigarrow F = S^1$

$$\begin{array}{ccc} S^1 & \to & S^3 = E_3 \\ & & \downarrow \\ & & S^2 \end{array}$$

Hopf fibration.

In this example, $E_3 \to E_2$ is a principal bundle.

In general, if the local coefficient system is trivial, then the maps $E_n \to E_{n-1}$ are all 'principal fibrations'. Just as with principal bundles, principal fibrations have classifying spaces.

$$\begin{array}{ccc} K(G,n-2) & \to & E_n \\ & & \downarrow \\ & & E_{n-1} \end{array}$$

is classified by a map $E_{n-1} \to BK(G,n-2)$.

Recall $\pi_n(BG) \cong \pi_{n-1}(G)$. It follows that $BK(G,n-2) = K(G,n-1)$.

$E_{n-1} \to K(G,n-1)$ classifying $E_n \to E_{n-1}$.

$$\begin{array}{ccc} [X, K(G,m)] & \to & H^m(X;G) \\ [f] & \longmapsto & f^* i_m \end{array} \qquad i_m \in H^m(K(G,m);G)$$

Natural transformation.

$$\begin{tikzcd}
 & X \arrow[d] & \\
 & \vdots \arrow[d] & \\
 & E_3 \arrow[r] \arrow[d] & K(G_3, 3) \\
 & E_2 \arrow[r] \arrow[d] & K(G_2, 2) \\
 & E_1 \arrow[r] \arrow[d] & K(G_1, 1) \\
Z \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuur, dotted] & Y &
\end{tikzcd}$$

$Z \to E_2$ lifts to $Z \to E_3$ if and only if the composition $Z \to E_2 \to K(G_2, 2)$ is nullhomotopic, i.e. the corresponding cohomology class in $H^2(Z; G_2)$ is zero.

We can study the Moore-Postnikov tower for $B\rho : BH \to BG$

$$\begin{tikzcd}
BH \arrow[d] & \\
\vdots \arrow[d] & \\
E_2 \arrow[r, "k_2"] \arrow[d] & K(G_2, 2) \\
E_1 \arrow[r, "k_1"] \arrow[d] & K(G_1, 1) \\
BG &
\end{tikzcd}$$

$k_n : E_n \to K(G_n, n)$ corresponds to a class $k_n \in H^n(E_n; G_n)$ which vanishes when pulled back to BH. Sometimes called 'Postnikov invariants of f'.

These are exactly the 'minimal set' of universal classes we desired. The pullback of these classes are the obstruction classes.