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Homogeneous geometry + Einstein metrics

Part I

Klein's view of geometry

Space $\leftarrow$ $(M, \underline{G})$ $\rightarrow$ group (Lie group)

Lie groups

Def G Lie group

a) Abstract group
b) Smooth mfd
c) Operations $m: G \times G \to G$ mult
$i: G \to G$ inverse
are smooth.

e.g. $(\mathbb{R}^n, +)$, $(\mathbb{C}^n, +)$, $(\mathbb{H}^n, +)$,
$M_n\mathbb{R} \cong \mathbb{R}^{n^2}$, $Gl_n\mathbb{R}$, $Gl_n\mathbb{C}$

Def 1) A Lie subgroup H of a Lie group G is

a) H a subgroup of G
b) H is an immersed submanifold of G
c) Group operations of H are smooth.

2) A closed subgroup of G is a closed subset of G.

Thm If H is a ~~closed~~ subgroup of a Lie group G, then H is a regular submfd of G, in particular it is a Lie subgroup of G.

e.g $Sl_n\mathbb{R}$, $O(n)$, $SO(n)$, $U(n)$, $SU(n)$

Matrix groups

e.g Lie groups not matrix groups: Heisenberg group.

The tangent space of a Lie group

G Lie group

$a \in G$, $L_a, R_a : G \to G$ Diffeom. $L_a(x) = a.x$, $R_a(x) = xa$

Hence $(dL_a)_e : T_eG \to T_aG$ isomorp

e.g $T_I Gl_n\mathbb{R} \cong M_n\mathbb{R} \cong \mathbb{R}^{n^2}$

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$T_I Sl_n\mathbb{R} \cong \{X \in M_n\mathbb{R} : \operatorname{tr} X = 0\}$
$= sl_n\mathbb{R}$

$T_I O(n) \cong \{X \in M_n\mathbb{R} : X^t = -X\}$
$= o(n)$

$T_I Aut(\underset{\text{v. space}}{V}) \cong T_I Gl_n\mathbb{R} \cong M_n\mathbb{R}$
$\cong End(V)$

The Lie algebra of a Lie group

Def A vector field X on G is called <u>left-invariant</u> if

$X \circ L_a = dL_a(X)$ i.e.

$$\begin{tikzcd} TG \arrow[r, "dL_a"] & TG \\ G \arrow[u, "X"] \arrow[r, "L_a"'] & G \arrow[u] \end{tikzcd}$$

$X_{ag} = (dL_a)_g(X_g)$
$a, g \in G$.

Left-inv. vector fields are smooth and are determined by their values at the identity $e \in G$

i.e. $X_a = \underline{(dL_a)_e}(X_e)$

Let $\mathfrak{g} = \{$left-inv. vector fields on $G\}$

- $\mathfrak{g}$ is $\mathbb{R}$-vector space (a subspace of $\mathcal{X}(G)$)
- $\mathfrak{g}$ is closed under bracket of v. fields (i.e. $\forall\, X, Y \in \mathfrak{g} \Rightarrow [X, Y] \in \mathfrak{g}$)

Hence $\mathfrak{g}$ is a Lie algebra called the Lie algebra of Lie group G

e.g. $SL_n\mathbb{R} \rightsquigarrow \mathfrak{sl}_n\mathbb{R}$, etc.

Prop $\mathfrak{g} \cong T_e G$ (isom. of v. spaces)

$$X \longmapsto X_e$$

$$X^v \longleftarrow v \in$$

$$X^v_g = (dL_g)_e(v), \quad g \in G$$

$$X_g = (dL_g)_e(v), \quad g \in G$$

left-inv.

The Lie bracket of $\mathfrak{g}$ induces a Lie bracket on T_eG:

$u, v \in T_e G$, $\quad [u,v] \overset{\text{def}}{=} [X^u, X^v]_e$

e.g. $G = Gl_n \mathbb{R}$

$$T_g Gl_n \mathbb{R} \simeq M_n \mathbb{R}$$

$$\sum a_{ij} \frac{\partial}{\partial x^{ij}}\Big|_g \longleftrightarrow (a_{ij})$$

If $B = \sum b_{ij} \frac{\partial}{\partial x^{ij}}\Big|_I \in T_I Gl_n \mathbb{R}$

and X^B is corresponding left-inv. v.f., then

$$X^B_g = \sum_{i,j} \left(\sum_k g_{ik} b_{kj} \right) \frac{\partial}{\partial x^{ij}}\Big|_g$$

$\forall\, g \in Gl_n \mathbb{R}$.

It follows that the bracket in $\mathfrak{gl}_n\mathbb{R} = T_I Gl_n\mathbb{R}$ is given by

$$[A, B] = AB - BA.$$

Relation between Lie groups and Lie algebras

1) If $\varphi: G \to H$ homom. of Lie groups (φ smooth + group homo)

then $d\varphi_e : T_e G \to T_{\varphi(e)} H$ is a Lie algebra homo

2) If G Lie group with Lie alg. $\mathfrak{g}$ and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$, then $\exists!$ connected subgroup H of G s.t. $\mathrm{Lie}(H) = \mathfrak{h}$

3) If G, H Lie groups

G simply connected

with $\mathfrak{g}, \mathfrak{h}$ Lie alg[illegible] with $\mathfrak{g}, \mathfrak{h}$ Lie algebras resp.

then for every homo $\psi : \mathfrak{g} \to \mathfrak{h}$

$\exists!$ Lie group homo $\varphi : G \to H$

s.t. $d\varphi_e = \psi$.

Lie's Theorems

1) For every Lie algebra $\mathfrak{g}$
there exists a Lie group G
s.t. $\mathrm{Lie}(G) \cong \mathfrak{g}$.

2) For G_1, G_2 Lie groups with
corresp. $\mathfrak{g}_1, \mathfrak{g}_2$ Lie algebras,
if $\mathfrak{g}_1 \cong \mathfrak{g}_2$ then G_1 and G_2
are locally isomorphic

If G_1, G_2 are simply connected then
$G_1 \cong G_2$

One-parameter subgroups

G Lie group

Def A smooth homo $\varphi : (\mathbb{R}, +) \to G$

is called a 1-parameter subgroup of G

e.g. $\varphi : \mathbb{R} \to Gl_3 \mathbb{R}$

$$\varphi(t) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^t \end{pmatrix}$$

($\varphi(\mathbb{R})$ is not a subgroup of G in general!)

thm There is a 1-1 corresp.

between 1-param. subgroups of G and $T_e G$

$\Rightarrow)$ For $\varphi : \mathbb{R} \to G$, then

$$d\varphi_0 \left(\frac{d}{dt}\Big|_0 \right) \in T_e G$$

$\Leftarrow)$ For $v \in T_e G$, let

$$X^v_g = (dL_g)_e (v)$$ corr left-inv vector field

$g \in G$

$X_g = (a_g)_e(v)$ vector field

Let $\varphi : \overset{=(-\varepsilon,\varepsilon)}{I} \to G$ be the unique integral curve of X^v

s.t. $\varphi(0) = e$, $\varphi'(t) = X^v_{\varphi(t)}$

Then φ is a group homo which extends to $\mathbb{R}$ by using group operation of G.

So obtain $\varphi^v : \mathbb{R} \to G$

__Corollary__ For every left-inv. v.f $X \in \mathfrak{g}$

$\exists!$ 1-param. subgroup $\varphi_X : \mathbb{R} \to G$

s.t. $\varphi_X'(0) = X$

__Def__ The exponential map

$$\exp : \mathfrak{g} \to G$$

$$\exp(X) = \varphi_X(1)$$

Γ $G = Gl_n\mathbb{R}$, $\exp : M_n \to Gl_n\mathbb{R}$

$\exp(X) = e^X$

$\exp(X) = e^X$ ⌋

Prop If G is abelian $\exp$ is onto

(need Campbell - Baker - Hausdorff formula)

Tools to study structure of Lie groups

A) the adjoint representation

B) Maximal tori

A) G Lie group

$$I_g = R_{g^{-1}} \circ L_g : G \to G$$

$$I_g(x) = g x g^{-1}$$ inner autom. of G

Def The <u>adjoint representation of G</u>

is the homo

$$Ad : G \longrightarrow Aut(\mathfrak{g})$$

$$Ad(g) = (dI_g)_e$$

<u>Def</u> the <u>adjoint representation</u> of $\mathfrak{g}$ is the homo

$$ad : \mathfrak{g} \longrightarrow End(\mathfrak{g})$$

defined by $\quad ad = d(Ad)_e$

<u>Prop</u> For all $X, Y \in \mathfrak{g}$

$$ad(X)(Y) = [X, Y]$$

$(ad X : \mathfrak{g} \to \mathfrak{g})$

<u>Prop</u> If G is a matrix group then $Ad(g)X = gXg^{-1}$, $g \in G$, $X \in \mathfrak{g}$

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the Killing form

The Killing form

Def The <u>Killing</u> form of a Lie algebra $\mathfrak{g}$ is the symmetric billinear form

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$$

$$B(X,Y) = \mathrm{tr}(\operatorname{ad} X \circ \operatorname{ad} Y)$$

Properties

- B is Ad-invariant i.e

$$B(X,Y) = B(\operatorname{Ad}(g)X, \operatorname{Ad}(g)Y)$$

$$g \in G, \; X, Y \in \mathfrak{g}$$

- $$B([X,Z],Y) = B(X,[Z,Y])$$

i.e. $\operatorname{ad} Z : \mathfrak{g} \to \mathfrak{g}$ is B-antisymmetric.

Def G <u>semi-simple</u> if

B is non-degenerate

B is non-degenerate

Thm G compact, semi-simple, then B is negative-definite.

Ex $su(n)$: $B(X,Y) = 2n \operatorname{tr} XY$

$so(n)$: $B(X,Y) = (n-2) \operatorname{tr} XY$

⋮

Maximal tori

Def An n-torus in G is $T^n \cong S^1 \times \cdots \times S^1$ subgroup

A torus T is called maximal if $\forall$ torus S in G with $T \subset S \subset G$ then $T = S$

Facts

1) Any torus is contained in a max torus

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2) If G cpt, then a max torus is a maximal connected abelian subgroup of G

3) Two maximal tori T_1, T_2 are conjugate

i.e. $g T_1 g^{-1} = T_2$ for some $g \in G$

<u>Def</u> The <u>rank</u> of a Lie group is the dimension of a max torus.

<u>e.g.</u> $G = U(n)$

$$T = \left\{ \begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{pmatrix} \right\} \simeq \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}$$

$\operatorname{rk} U(n) = n = \operatorname{rk} SO(n) = \operatorname{rk} SO(2n+1)$
$= \operatorname{rk} Sp(n)$

$\operatorname{rk} SU(n) = n-1$

Classification Theorem of compact, connected Lie groups

A Lie group is called <u>simple</u> if it is not abelian and it does not contain a <u>proper</u> normal Lie subgroup

<u>Thm</u> 1) If G <u>cpt</u>, <u>conn.</u> Lie group then there is a covering space $\tilde{G}$ isomor to $T^k \times G'$,

$$G' = \underline{\text{cpt}},\ \underline{\text{conn}},\ \underline{\text{simply conn}}.$$

2) Every cpt, conn, simply conn Lie group is isom to the product of <u>simple</u>, <u>cpt</u>, <u>conn</u>, <u>simpl. conn.</u> Lie groups

3) The <u>simple</u> <u>cpt</u>, <u>conn</u>, <u>simply conn</u> Lie groups are

Lie groups are

$SU(n)$, $n \geq 2$, $\widetilde{SO}(2n+1)$ B_n $n \geq 3$, $Sp(n)$ C_n $n \geq 2$

A_{n-1} $\widetilde{SO}(2n)$ D_n $n \geq 4$

$\widetilde{SO} \equiv \mathrm{Spin}$)

G_2, F_4, E_6, E_7, E_8 (exceptional Lie groups)