22-3-24
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Homogeneous spaces

Def A manifold M is homogeneous if a Lie Group G acts transitively on M
$G \times M \to M \quad (g,p) \mapsto g \cdot p$

For $p \in M$, $G_p = \{ g \in G \quad g \cdot p = p \}$ isotropy subgroup of G

The orbit of $p \in M$ is $G \cdot p = \{ g \cdot m \mid g \in G \} \subset M$

Thm Let G acts transitivd of M and $H = G_p$ isotropy at $p \in M$
then
1) G_p is closed in G
2) The map $\boxed{G/H} \to M \quad gH \to g \cdot p$ is diffeom
3) $G \cdot p \cong G/H$ diffeom

In general, If G Lie group, H closed sb of G
Then G/H admits a mfd structure
s.t. $\pi : G \to G/H \quad \pi(g) = gH$
is a submersion (i.e $d\pi_e$ is onto)

Then G acts on G/H by
$(g, aH) \longmapsto (ga)H \quad , g, a \in G$

A homog Riemannian mfd M is
a Riemannian mfd (M, g)

in which $I(M)$ acts transitively

$\underbrace{I(M)}$ isometry group of $M \to$ Lie group

Klein geometry (G, H)

(Lie, closed)

<u>Note</u> $M = G/H = G'/H'$

Examples

1) $G = G/\{e\} = G \times G / G$

$(G \times G) \times G \to G$

$(g_1, g_2) \cdot g = g_1 g g_2^{-1}$

2) $O(n+1)$ acts on $S^n \subset \mathbb{R}^{n+1}$

$S^n \cong O(n+1)/O(n) \cong \dfrac{SO(n+1)}{SO(n)}$

$S^{2n+1} \cong U(n+1)/U(n) \cong \dfrac{SU(n+1)}{SU(n)}$

3) $\mathbb{RP}^n \cong SO(n+1)/O(n)$

$\mathbb{CP}^n \cong SU(n+1)/U(n)$

4) Grassmann mflds

$Gr_k \mathbb{R}^n = \{k\text{-dim subsp in } \mathbb{R}^n\}$

$\cong O(n)/O(k) \times O(n-k)$

5) Stiefel mflds

$V_k \mathbb{R}^n = \{k\text{-frames in } \mathbb{R}^n\} \cong \dfrac{O(n)}{O(n-k)}$

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k o.n. vectors in $\mathbb{R}^n$

6) Full flag mfld

$$F_n(\mathbb{C}) = \{ f \mid f \text{ flag in } \mathbb{C}^n \}$$

$$f = \{ V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n \subset \mathbb{C}^n \}$$

$U(n)$ acts on $F_n(\mathbb{C})$

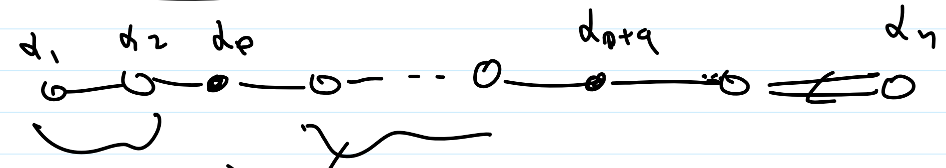
$$F_n(\mathbb{C}) \cong U(n) / U(1) \times \cdots \times U(1) \cong \frac{U(n)}{T^{max}}$$

Generalized flag mfds

$$F = SU(n) / S(U(n_1) \times \cdots \times U(n_s)) = \frac{SU(n)}{\underbrace{C(S)}_{\text{centralizer of a torus}}}$$

$$\boxed{M = G / C(S)}$$

$Sp(n)$



$$Sp(n) / U(p) \times U(q) \times Sp(n-p-q)$$

Geometry $M = G/H$

$a \in G$ $\quad \tau_a : G/H \longrightarrow G/H$

$$\tau_a(gH) = agH$$

Def A Riem metric g on G/H is called $\underline{G\text{-inv}}$ if τ_a is an isometry

Also $\pi : G \to G/H$ such

$$d\pi_e : \mathfrak{g} \to T_o(G/H) \text{ onto}$$

to $\underline{\ker d\pi_e = \mathfrak{h}}$ so $\boxed{\mathfrak{g}/\mathfrak{h} \cong T_o(G/H)}$

Def G/H is called $\underline{\text{reductive}}$

if $\exists$ subspace $\mathfrak{m} \subset \mathfrak{g}$ st

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

and $\cdot \, Ad(h)\mathfrak{m} \subset \mathfrak{m} \quad \forall h \in H$

i e $\mathfrak{m}$ is $Ad(H)$-inv.

Def The $\underline{\text{isotropy representation}}$ of G/H is the smooth homo

$$\chi = Ad^{G/H} : H \to Aut(T_o(G/H))$$

$$\chi(h)(X) = (d\tau_h)_o(X)$$

If G/H is reductive $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$

$$\chi : H \to Aut(\mathfrak{m})$$

Prop ...

Prop there is 1–1 corresp between

1) G-inv. Riem metrics on G/H

2) $\mathrm{Ad}^{G/H}$-inv inner product $\langle , \rangle$ on $\mathfrak{m}$

i.e $\langle X, Y \rangle = \langle \mathrm{Ad}^{G/H}(h)X, \mathrm{Ad}^{G/H}(h)Y \rangle$

If $H = \{e\}$ then $\mathfrak{g} = T_e G$

G-inv metrics are left-inv metric on G.

The isotropy repr of a reductive homog space satisfies:

$$\mathrm{Ad}^G : G \to \mathrm{Aut}(\mathfrak{g})$$

$$\mathrm{Ad}^H : H \to \mathrm{Aut}(\mathfrak{h})$$

$$\mathrm{Ad}^{G/H} : H \to \mathrm{Aut}(\mathfrak{m})$$

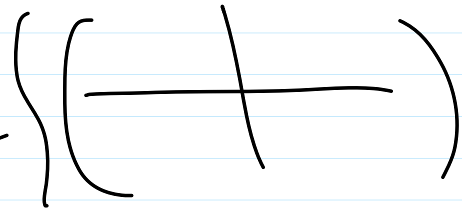
$$\mathrm{Ad}^G \big|_H = \mathrm{Ad}^H \oplus \mathrm{Ad}^{G/H}$$

e.g $G/H = V_2\mathbb{R}^4 = SO(4)/SO(2)$

$\mathrm{Ad}^{G/H} = 1 \oplus \lambda_2 \oplus \lambda_2$, $\lambda_2 : SO(2) \to SO(2)$ stand. repr of $SO(2)$

trivial

Then $\underset{5}{\mathfrak{m}} = \underset{1}{\mathfrak{m}_0} \oplus \underset{2}{\mathfrak{m}_1} \oplus \underset{2}{\mathfrak{m}_2}$

$\mathfrak{m} =$ 

Def A G-inv metric g on G/H ($\leftrightarrow \langle , \rangle$)

is called <u>naturally reductive</u>

if $\langle [X,Y]_{\mathfrak{m}}, Z \rangle = \langle X, [Y,Z]_{\mathfrak{m}} \rangle$

$\forall\ X, Y, Z \in \mathfrak{m}$ $\quad (\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m})$

<u>Note about G-inv metrics</u>

Let $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_\ell$, $\mathrm{Ad}(H)$-modules

If $\mathfrak{m}_i \not\cong \mathfrak{m}_j$ then

any G-inv metric is

$\langle , \rangle = x_1(-B)\big|_{\mathfrak{m}_1} + \cdots + x_\ell(-B)\big|_{\mathfrak{m}_\ell}$, $x_i > 0$

<u>Curvature</u> (eg. Besse)

$M = G/H$, $\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{m}$

$\mathfrak{m} \cong T_o(G/H)$, $o = eH$

$$X \longmapsto X^*_o = \frac{d}{dt}(\exp tX)\cdot o \Big|_{t=0}$$

X^* is a Killing v.f on M

$$\mathcal{L}_{X^*}\langle Y, Z\rangle = \langle [X^*, Y], Z\rangle + \langle Y, [X^*, Z]\rangle$$

Levi-Civita connection

$$\nabla_{X^*} Y^* \Big|_0 = -\frac{1}{2}[X, Y] + U(X, Y)$$

$$U : m \times m \longrightarrow m$$

$$2\langle U(X,Y), Z\rangle = \langle [Z, X]_m, Y\rangle + \langle X, [Z, Y]_m\rangle$$

If $U \equiv 0$ G/H is nat reductive

Curvature tensor

$$\langle R(X,Y)X, Y\rangle = -\frac{3}{4}\|[X,Y]_m\|^2 - \frac{1}{2}\langle \quad \rangle - - -$$

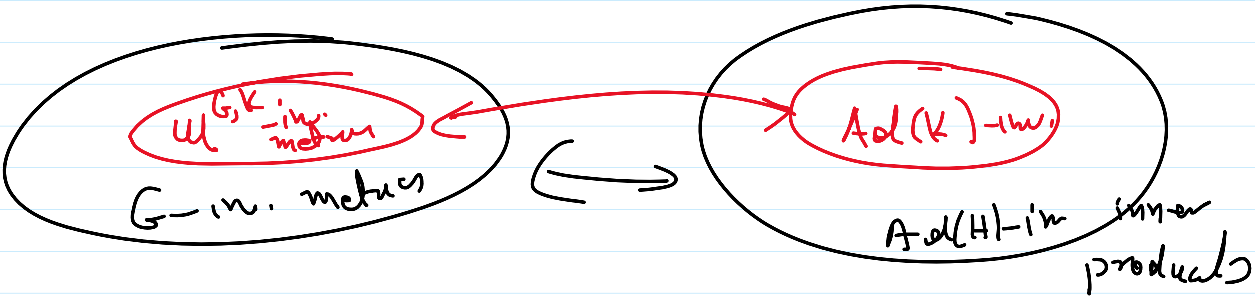
Ricci tensor, Scalar curvature

Non-diagonal metrics

e.g $\mathbb{R}^4 = G/H$

e.g. $V_k\mathbb{R}^n = G/H$

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \cdots \oplus \mathfrak{m}_\ell$$



$$G \subset H$$

$$H \subset K \subset G \quad \text{with} \quad \underbrace{K \subset N_G(H)}$$

n.x

$$G/H = SO(k_1+k_2+k_3)/SO(k_3)$$

$$K = SO(k_1)\times SO(k_2)\times SO(k_3)$$

$$\underbrace{K/H}_{\hat{\mathfrak{g}}} \longrightarrow G/H \longrightarrow \underbrace{G/K}_{\check{\mathfrak{g}}}$$

$$\mathfrak{g} = \check{\mathfrak{g}} + \hat{\mathfrak{g}}$$