Introduction to Graph Complexes

Ben Ward

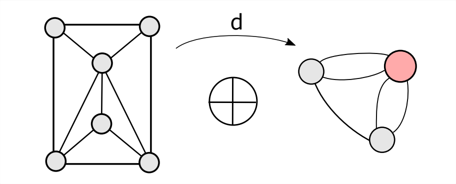
Bowling Green State University

IISER – Kolkata
November 2025

An early campus morning...

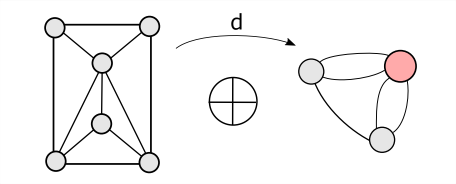
An early campus morning...

Overview



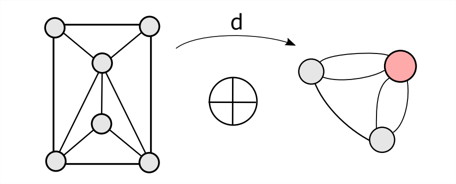
1. Use graphs to compute invariants of certain topological spaces.

Overview



1. Use graphs to compute invariants of certain topological spaces.
2. There are many variations on this story.

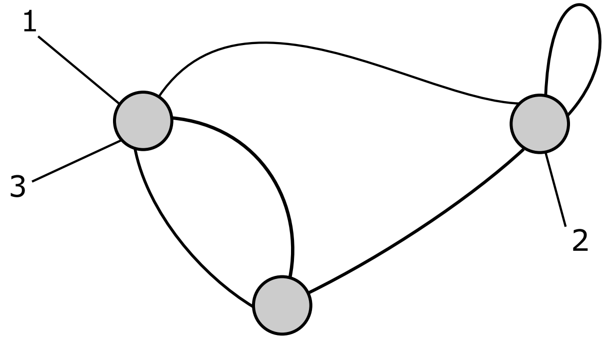
Overview



1. Use graphs to compute invariants of certain topological spaces.
2. There are many variations on this story.
3. Studying all the variations together gives more information than studying them individually.

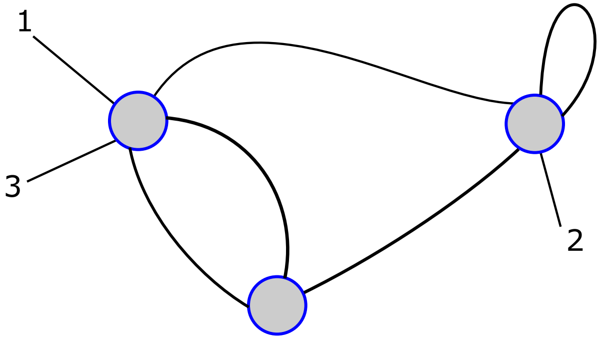
Here is a graph

It has...



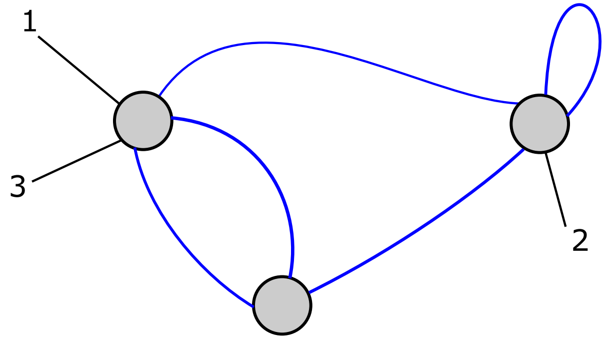
Here is a graph

It has... a set of vertices.



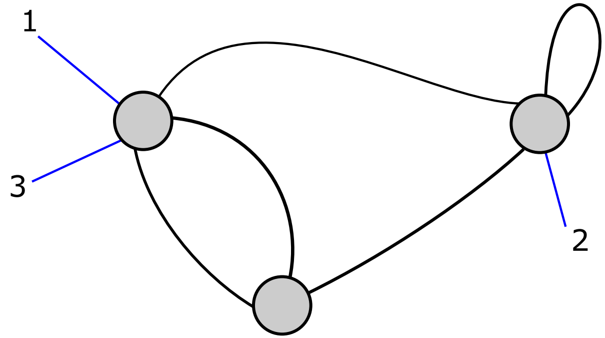
Here is a graph

It has...a set of edges.



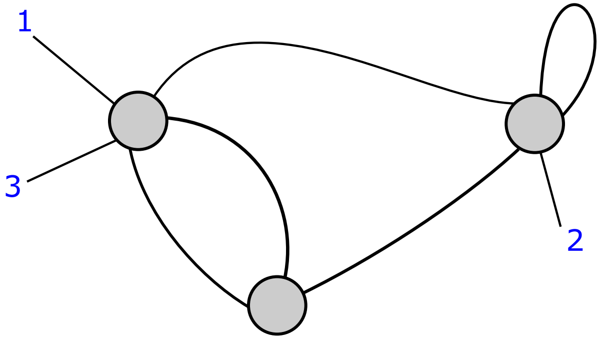
Here is a graph

It has...a set of legs,



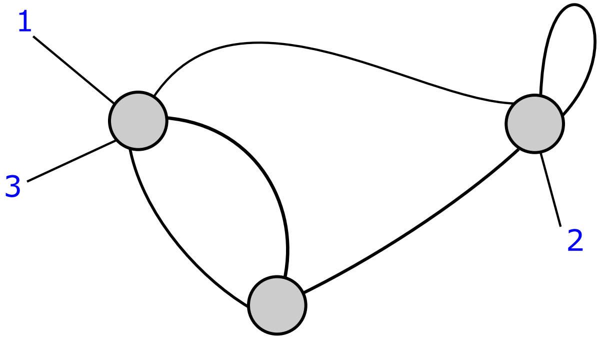
Here is a graph

It has...legs which are numbered.



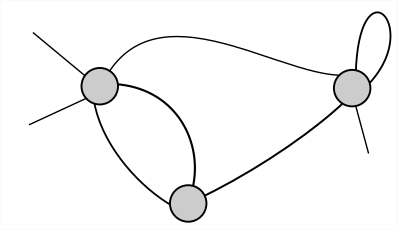
Here is a graph

It is connected. It is not planar.



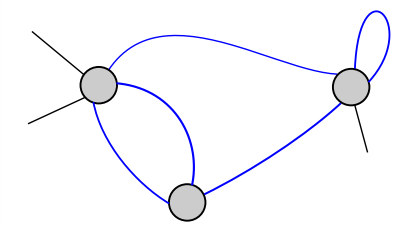
Vector space associated to a graph

Let γ be a graph.



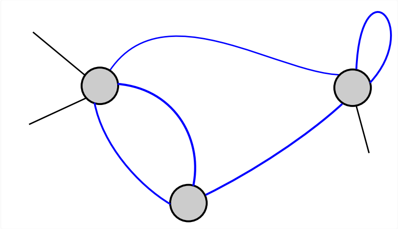
Vector space associated to a graph

Let γ be a graph. Let $E(\gamma)$ be its set of edges.



Vector space associated to a graph

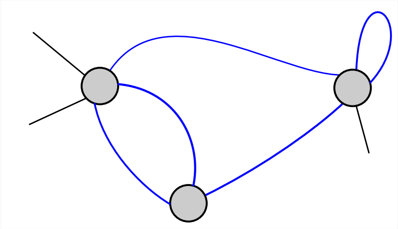
Let γ be a graph. Let $E(\gamma)$ be its set of edges.



Define $\det(\gamma) = \ldots.$

Vector space associated to a graph

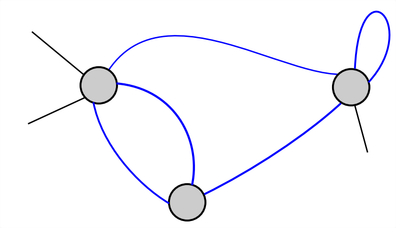
Let γ be a graph. Let $E(\gamma)$ be its set of edges.



Define $\det(\gamma) = \operatorname{span}_{\mathbb{Q}}(E(\gamma)) \otimes_{S_n} sgn_n$

Vector space associated to a graph

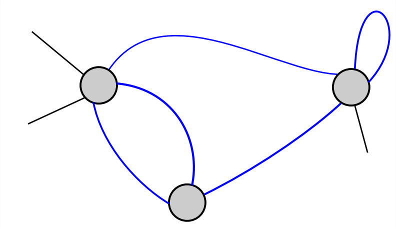
Let γ be a graph. Let $E(\gamma)$ be its set of edges.



Define $\det(\gamma) = \ \mathrm{span}_{\mathbb{Q}}(E(\gamma)) \otimes_{S_n} sgn_n \cong \mathbb{Q}$.

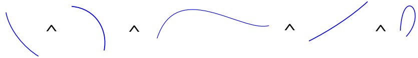
Vector space associated to a graph

Let γ be a graph. Let $E(\gamma)$ be its set of edges.



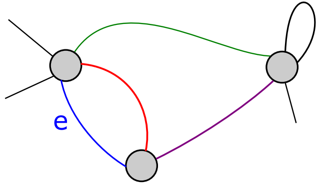
Define $\det(\gamma) = \text{span}_{\mathbb{Q}}(E(\gamma)) \otimes_{S_n} sgn_n \cong \mathbb{Q}$.

Here is a vector in $\det(\gamma)$:



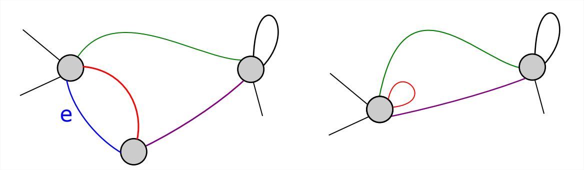
Edge Contraction

Let γ be a graph. Let e be an edge.



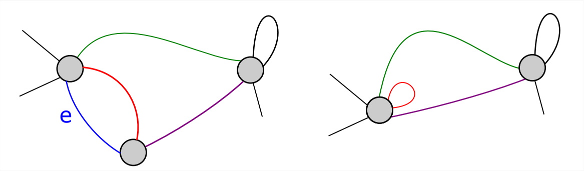
Edge Contraction

Let γ be a graph. Let e be an edge. Form γ/e by contracting e.



Edge Contraction

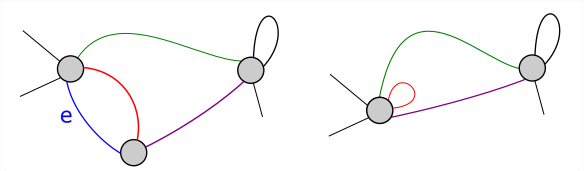
Let γ be a graph. Let e be an edge. Form γ/e by contracting e.



Define $d_e\colon \det(\gamma) \to \det(\gamma/e)$ by $d_e(w \wedge e) = w$.

Edge Contraction

Let γ be a graph. Let e be an edge. Form γ/e by contracting e.

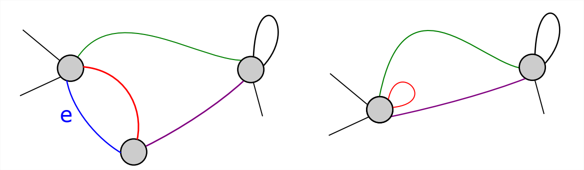


Define $d_e \colon \det(\gamma) \to \det(\gamma/e)$ by $d_e(w \wedge e) = w$.
In particular

$$d_e d_{e'} = -d_{e'} d_e$$

Edge Contraction

Let γ be a graph. Let e be an edge. Form γ/e by contracting e.



Define $d_e \colon \det(\gamma) \to \det(\gamma/e)$ by $d_e(w \wedge e) = w$.
In particular

$$d_e d_{e'} = -d_{e'} d_e$$

Hence $d := \sum_e d_e$ satisfies $d^2 = 0$.

Graph complex

Define a chain complex

Graph complex

Define a chain complex

$$\mathsf{GC} = \bigoplus_{\gamma} \ldots$$

Graph complex

Define a chain complex

$$\mathsf{GC} = \bigoplus_{\gamma} det(\gamma)...$$

Graph complex

Define a chain complex

$$\mathrm{GC} = \left(\bigoplus_{\gamma} det(\gamma) \right) / Iso$$

Graph complex

Define a chain complex

$$\mathsf{GC} = \left(\bigoplus_{\gamma} det(\gamma) \right) / Iso \cong \bigoplus_{[\gamma]} det(\gamma)_{Aut(\gamma)}$$

with differential induced by $d = \sum_e d_e$.

Graph complex

Define a chain complex

$$\mathrm{GC} = \left(\bigoplus_{\gamma} det(\gamma) \right) / Iso \cong \bigoplus_{[\gamma]} det(\gamma)_{Aut(\gamma)}$$

with differential induced by $d = \sum_e d_e$.

$$\dim(\det(\gamma)_{Aut(\gamma)}) = 0 \text{ or } 1$$

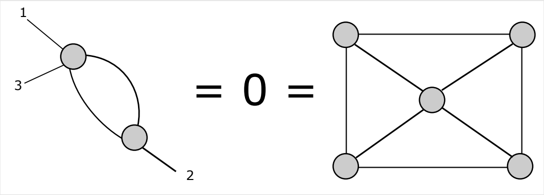
Graph complex

Define a chain complex

$$\mathrm{GC} = \left(\bigoplus_{\gamma} det(\gamma) \right) / Iso \cong \bigoplus_{[\gamma]} det(\gamma)_{Aut(\gamma)}$$

with differential induced by $d = \sum_e d_e$.

$$\dim(\det(\gamma)_{Aut(\gamma)}) = 0 \text{ or } 1$$



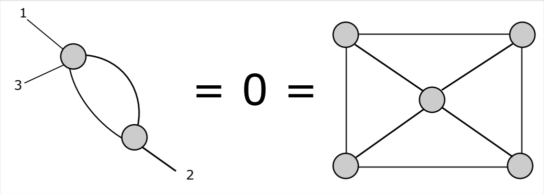
Graph complex

Define a chain complex

$$\mathrm{GC} = \left(\bigoplus_{\gamma} det(\gamma) \right) / Iso \cong \bigoplus_{[\gamma]} det(\gamma)_{Aut(\gamma)}$$

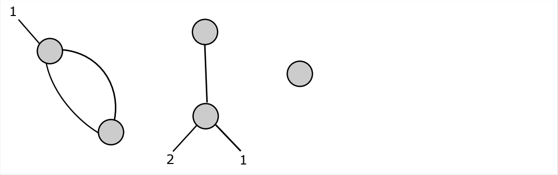
with differential induced by $d = \sum_e d_e$.

$$\dim(\det(\gamma)_{Aut(\gamma)}) = 0 \text{ or } 1$$

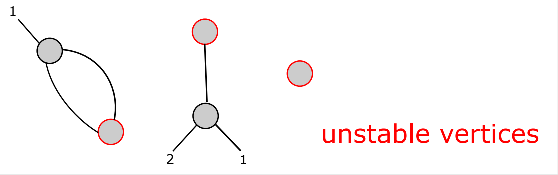


Question: what is the homology of this chain complex?

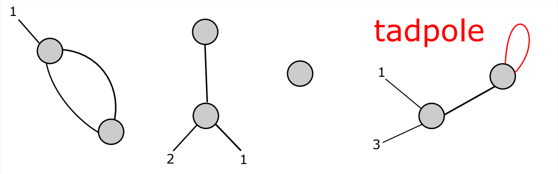
First a few simplifying assumption: graphs are stable and without tadpoles.



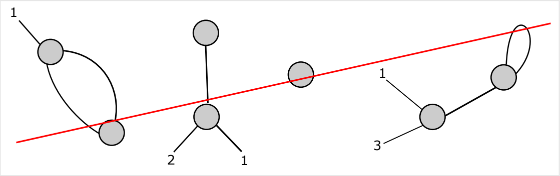
First a few simplifying assumption: graphs are stable and without tadpoles.



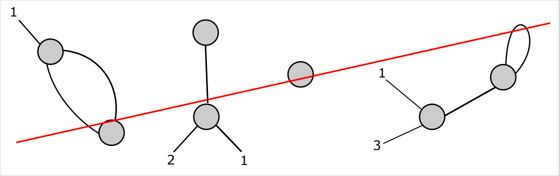
First a few simplifying assumption: graphs are stable and without tadpoles.



First a few simplifying assumption: graphs are stable and without tadpoles.

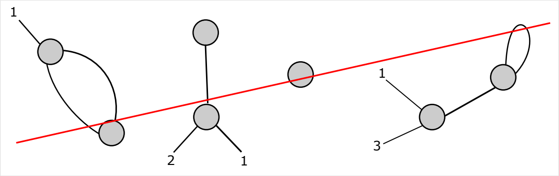


First a few simplifying assumption: graphs are stable and without tadpoles.



So d preserves n, the number of legs and the genus $g := |E| - |V| + 1$

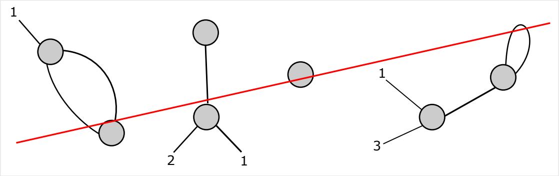
First a few simplifying assumption: graphs are stable and without tadpoles.



So d preserves n, the number of legs and the genus $g := |E| - |V| + 1$
Hence

$$\mathrm{GC} = \bigoplus_{g,n} \mathrm{GC}_{g,n}$$

First a few simplifying assumption: graphs are stable and without tadpoles.



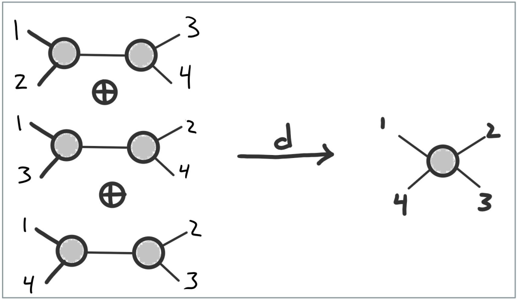
So d preserves n, the number of legs and the genus $g := |E| - |V| + 1$
Hence

$$\mathrm{GC} = \bigoplus_{g,n} \mathrm{GC}_{g,n}$$

and each $\mathrm{GC}_{g,n}$ is finite dimensional.

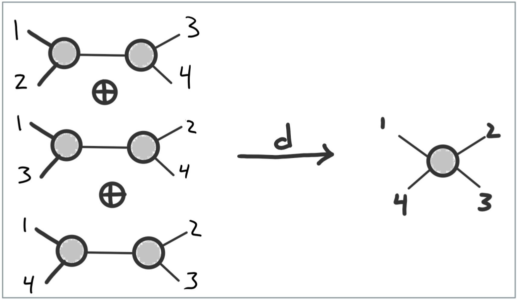
For example:

The chain complex: $\mathsf{GC}_{0,4}$



For example:

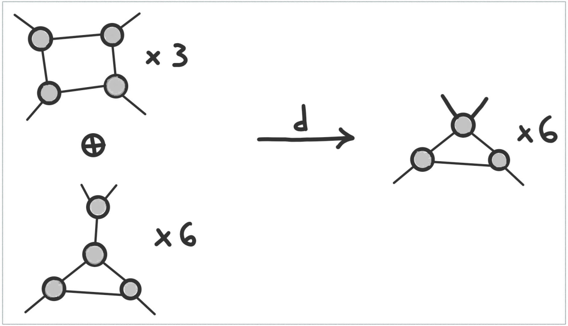
The chain complex: $\mathsf{GC}_{0,4}$



$$H_i(\mathsf{GC}_{0,4}) \cong \begin{cases} \mathbb{Q}^2 & \text{if } i = 1 \\ 0 & \text{else} \end{cases}$$

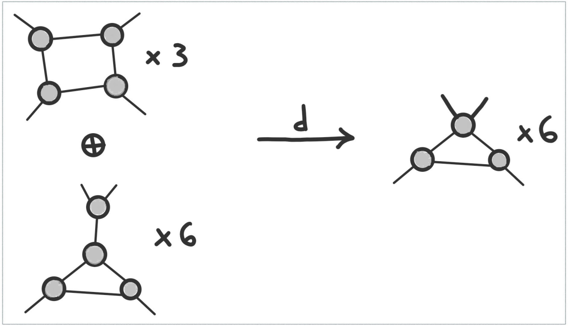
For example:

The chain complex: $\mathsf{GC}_{1,4}$



For example:

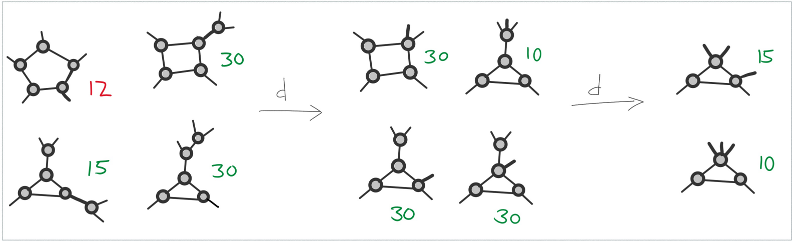
The chain complex: $\mathsf{GC}_{1,4}$



$$H_i(\mathsf{GC}_{1,4}) \cong \begin{cases} \mathbb{Q}^3 & \text{if } i = 4 \\ 0 & \text{else} \end{cases}$$

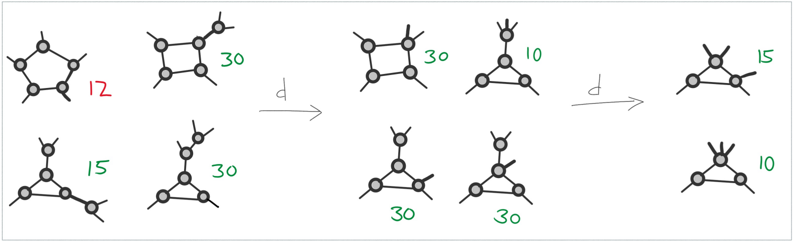
For example:

The chain complex: $\mathsf{GC}_{1,5}$



For example:

The chain complex: $\mathrm{GC}_{1,5}$



$$H_i(\mathrm{GC}_{1,4}) \cong \begin{cases} \mathbb{Q}^{12} & \text{if } i = 5 \\ 0 & \text{else} \end{cases}$$

A few first results

When $g = 0$ and $n \geq 3$, [Ginzburg-Kapranov '94]

A few first results

When $g = 0$ and $n \geq 3$, [Ginzburg-Kapranov '94]

$$\beta_i(\mathsf{GC}_{0,n}) = \begin{cases} (n-2)! & \text{if } i = n-3 \\ 0 & \text{else} \end{cases}$$

A few first results

When $g = 0$ and $n \geq 3$, [Ginzburg-Kapranov '94]

$$\beta_i(\mathsf{GC}_{0,n}) = \begin{cases} (n-2)! & \text{if } i = n-3 \\ 0 & \text{else} \end{cases}$$

When $g = 1$ and $n \geq 3$ [Chan-Galatius-Payne '22]

A few first results

When $g = 0$ and $n \geq 3$, [Ginzburg-Kapranov '94]

$$\beta_i(\mathsf{GC}_{0,n}) = \begin{cases} (n-2)! & \text{if } i = n-3 \\ 0 & \text{else} \end{cases}$$

When $g = 1$ and $n \geq 3$ [Chan-Galatius-Payne '22]

$$\beta_i(\mathsf{GC}_{1,n}) = \begin{cases} (n-1)!/2 & \text{if } i = n \\ 0 & \text{else} \end{cases}$$

A few first results

When $g=0$ and $n \geq 3$, [Ginzburg-Kapranov '94]

$$\beta_i(\mathrm{GC}_{0,n}) = \begin{cases} (n-2)! & \text{if } i = n-3 \\ 0 & \text{else} \end{cases}$$

When $g=1$ and $n \geq 3$ [Chan-Galatius-Payne '22]

$$\beta_i(\mathrm{GC}_{1,n}) = \begin{cases} (n-1)!/2 & \text{if } i = n \\ 0 & \text{else} \end{cases}$$

When $g=2$ and $n \geq 4$ [Chan '22]

$$\chi(\mathrm{GC}_{2,n}) = \pm \frac{n!}{12}$$

A few first results

When $g = 0$ and $n \geq 3$, [Ginzburg-Kapranov '94]

$$\beta_i(\mathsf{GC}_{0,n}) = \begin{cases} (n-2)! & \text{if } i = n-3 \\ 0 & \text{else} \end{cases}$$

When $g = 1$ and $n \geq 3$ [Chan-Galatius-Payne '22]

$$\beta_i(\mathsf{GC}_{1,n}) = \begin{cases} (n-1)!/2 & \text{if } i = n \\ 0 & \text{else} \end{cases}$$

When $g = 2$ and $n \geq 4$ [Chan '22]

$$\chi(\mathsf{GC}_{2,n}) = \pm \frac{n!}{12}$$

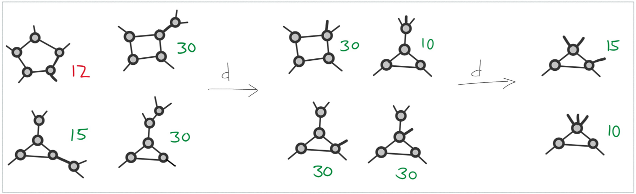
- The betti numbers are unknown!

Collapse of repeated markings

How to prove the previous results?

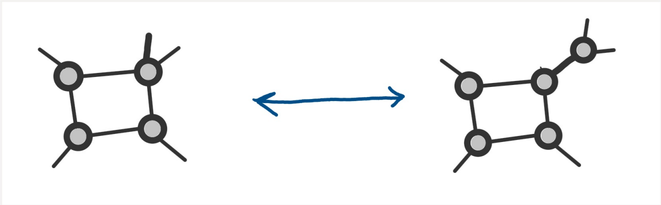
Collapse of repeated markings

How to prove the previous results? Recall the cancellation:



Collapse of repeated markings

How to prove the previous results? Recall the cancellation:

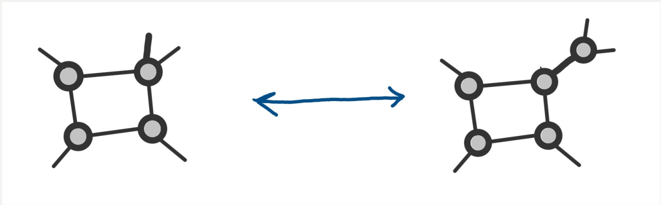


Collapse of repeated markings

How to prove the previous results? Recall the cancellation:

Collapse of repeated markings

How to prove the previous results? Recall the cancellation:



Theorem (CGP '22)

Let $g \geq 1$. The subcomplex of $\mathrm{GC}_{g,n}$ indexed by graphs with repeated markings is acyclic.

Why study graph homology?

Theorem (Willwacher '15)

$$\bigoplus H^{-2g}(\mathsf{GC}^*_{g,0}) \cong \mathfrak{grt}_1$$

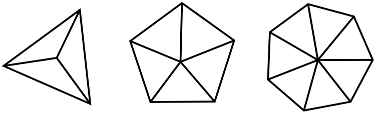
Why study graph homology?

Theorem (Willwacher '15)

$$\bigoplus H^{-2g}(\mathrm{GC}^*_{g,0}) \cong \mathfrak{grt}_1$$

Corollary (Willwacher)

Odd wheels are not boundaries.



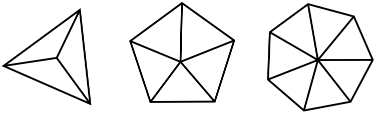
Why study graph homology?

Theorem (Willwacher '15)

$$\bigoplus H^{-2g}(\mathsf{GC}^*_{g,0}) \cong \mathfrak{grt}_1$$

Corollary (Willwacher)

Odd wheels are not boundaries.



Why study graph homology?

Theorem (Chan-Galatius-Payne '21 & '22)

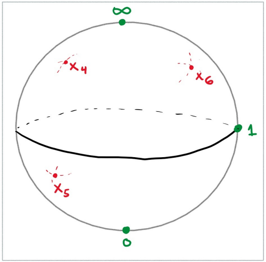
$$H_*(\mathsf{GC}_{g,n}) \hookrightarrow H^*(\mathcal{M}_{g,n})$$

Why study graph homology?

Theorem (Chan-Galatius-Payne '21 & '22)

$$H_*(\mathrm{GC}_{g,n}) \hookrightarrow H^*(\mathcal{M}_{g,n})$$

For example $\mathcal{M}_{0,n} := C(S^2, n)/\sim$ modulo Mobius transformations...

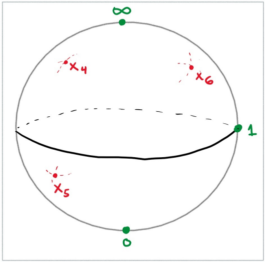


Why study graph homology?

Theorem (Chan-Galatius-Payne '21 & '22)

$$H_*(\mathrm{GC}_{g,n}) \hookrightarrow H^*(\mathcal{M}_{g,n})$$

For example $\mathcal{M}_{0,n} := C(S^2, n)/\sim$ modulo Mobius transformations...



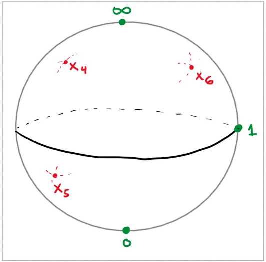
For example $H^{n-3}(\mathcal{M}_{0,n}) \cong H_{n-3}(\mathrm{GC}_{0,n})$.

Why study graph homology?

Theorem (Chan-Galatius-Payne '21 & '22)

$$H_*(\mathrm{GC}_{g,n}) \hookrightarrow H^*(\mathcal{M}_{g,n})$$

For example $\mathcal{M}_{0,n} := C(S^2, n)/\sim$ modulo Mobius transformations...



For example $H^{n-3}(\mathcal{M}_{0,n}) \cong H_{n-3}(\mathrm{GC}_{0,n})$.

Corollary

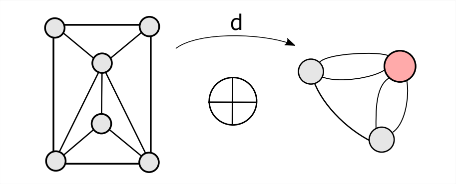
For g odd, $H^{4g-6}(\mathcal{M}_{g,0}) \neq 0$.

Transition

There are many variations on the graph complex construction...

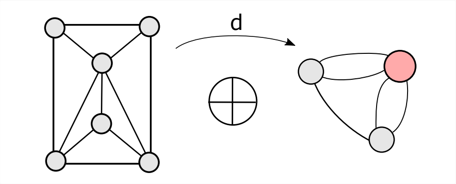
Transition

There are many variations on the graph complex construction...



Transition

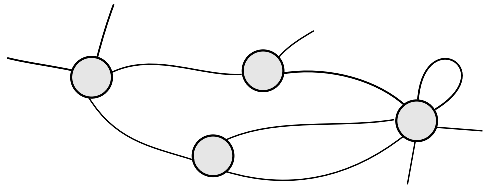
There are many variations on the graph complex construction...



... let me give you one.

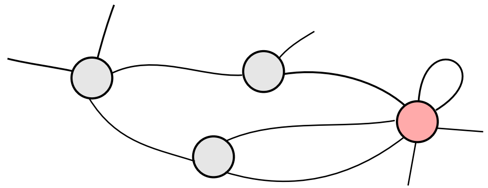
Here is a marked graph

Now, consider graphs with additional decorations:



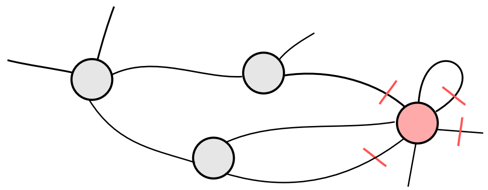
Here is a marked graph

Now, consider graphs with additional decorations:
the choice of a vertex.



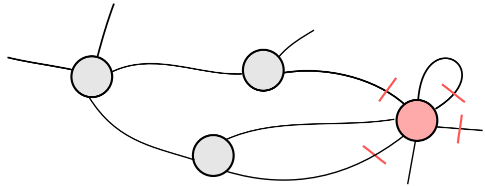
Here is a marked graph

Now, consider graphs with additional decorations:
the choice of a vertex and a subset of the half-edges adjacent to said vertex.



Here is a marked graph

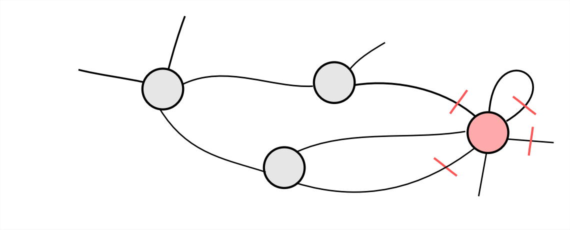
Now, consider graphs with additional decorations:
the choice of a vertex and a subset of the half-edges adjacent to said vertex.



A marked graph is type (g, n, r) provided

Here is a marked graph

Now, consider graphs with additional decorations:

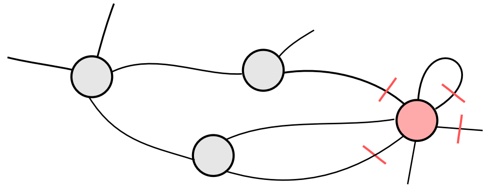


A marked graph is type (g, n, r) provided

- $g = \#\text{Edges} - \#\text{Vertices} + 2$
- $n = \#\text{legs}$
- $r = \#\text{marked half edges}$

Here is a marked graph

Now, consider graphs with additional decorations:



A marked graph is type (g, n, r) provided

- $g = \#\text{Edges} - \#\text{Vertices} + 2$
- $n = \#\text{legs}$
- $r = \#\text{marked half edges}$

This graph is of type $(4, 5, 4)$.

Chain complex of marked graphs: MGC

Fix $g, n \geq 0$ with $2g + n \geq 3$. Define

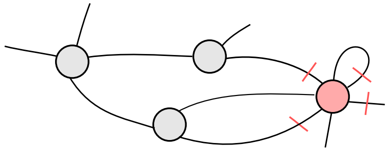
$$\mathrm{MGC}(g, n, r) = \bigoplus det(\gamma)_{Aut(\gamma)}$$

Chain complex of marked graphs: MGC

Fix $g, n \geq 0$ with $2g + n \geq 3$. Define

$$\mathrm{MGC}(g, n, r) = \bigoplus det(\gamma)_{Aut(\gamma)}$$

taken over isomorphism graphs of type (g, n, s) where $s \geq r$.
The differential:

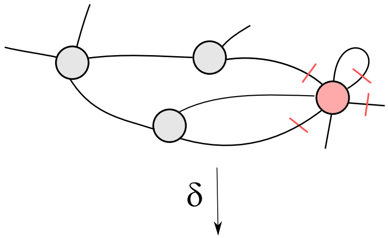


Chain complex of marked graphs: MGC

Fix $g, n \geq 0$ with $2g + n \geq 3$. Define

$$\mathrm{MGC}(g,n,r) = \bigoplus det(\gamma)_{Aut(\gamma)}$$

taken over isomorphism graphs of type (g, n, s) where $s \geq r$.
The differential: sum over edge contractions...

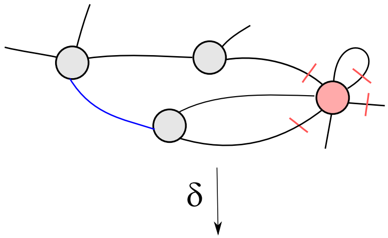


Chain complex of marked graphs: MGC

Fix $g, n \geq 0$ with $2g + n \geq 3$. Define

$$\mathrm{MGC}(g,n,r) = \bigoplus det(\gamma)_{Aut(\gamma)}$$

taken over isomorphism graphs of type (g, n, s) where $s \geq r$.
The differential: sum over edge contractions...

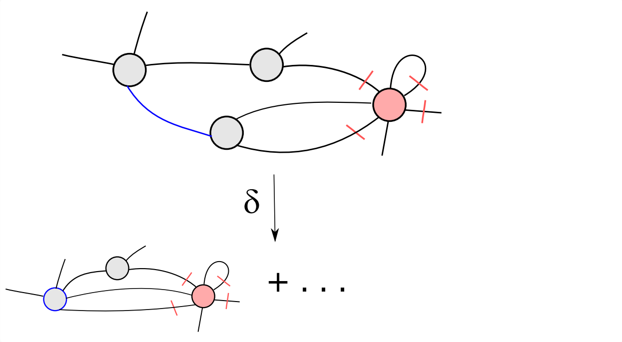


Chain complex of marked graphs: MGC

Fix $g, n \geq 0$ with $2g + n \geq 3$. Define

$$\mathrm{MGC}(g,n,r) = \bigoplus det(\gamma)_{Aut(\gamma)}$$

taken over isomorphism graphs of type (g, n, s) where $s \geq r$.
The differential: sum over edge contractions...

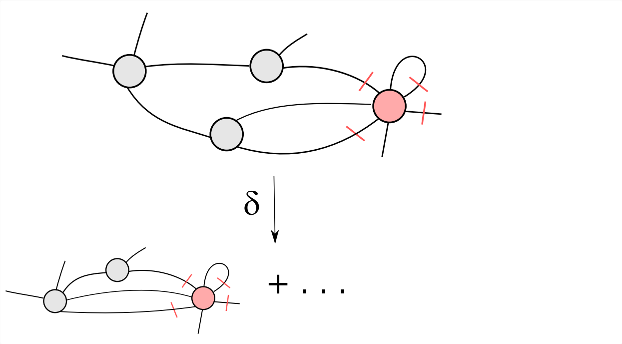


Chain complex of marked graphs: MGC

Fix $g, n \geq 0$ with $2g + n \geq 3$. Define

$$\mathrm{MGC}(g, n, r) = \bigoplus det(\gamma)_{Aut(\gamma)}$$

taken over isomorphism graphs of type (g, n, s) where $s \geq r$.
The differential: sum over edge contractions...

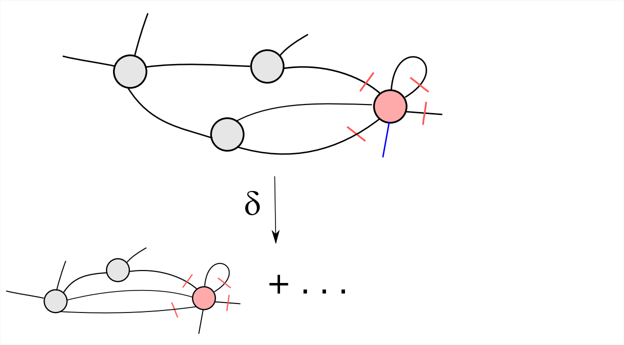


Chain complex of marked graphs: MGC

Fix $g, n \geq 0$ with $2g + n \geq 3$. Define

$$\mathrm{MGC}(g, n, r) = \bigoplus det(\gamma)_{Aut(\gamma)}$$

taken over isomorphism graphs of type (g, n, s) where $s \geq r$.
The differential: sum over edge contractions and ways to add a marking.

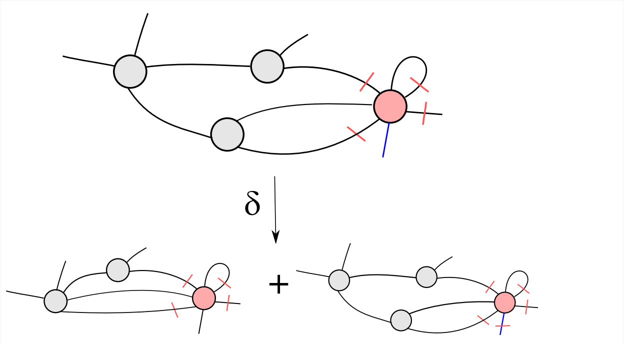


Chain complex of marked graphs: MGC

Fix $g, n \geq 0$ with $2g + n \geq 3$. Define

$$\mathrm{MGC}(g, n, r) = \bigoplus det(\gamma)_{Aut(\gamma)}$$

taken over isomorphism graphs of type (g, n, s) where $s \geq r$.
The differential: sum over edge contractions and ways to add a marking.

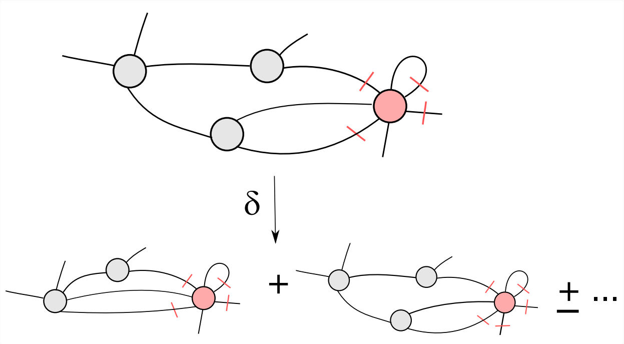


Chain complex of marked graphs: MGC

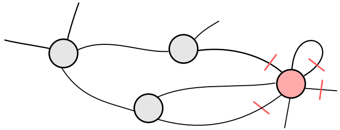
Fix $g, n \geq 0$ with $2g + n \geq 3$. Define

$$\mathrm{MGC}(g, n, r) = \bigoplus det(\gamma)_{Aut(\gamma)}$$

taken over isomorphism graphs of type (g, n, s) where $s \geq r$.
The differential: sum over edge contractions and ways to add a marking.



Why MGC(g, n, r)?

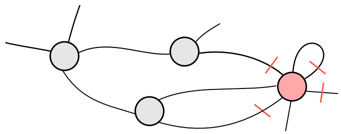


Theorem (Payne-Willwacher '24)

There is an injection

$$H_*(\mathrm{MGC}(g, n, 11)) \hookrightarrow H^*(\mathcal{M}_{g,n})$$

Why MGC(g, n, r)?



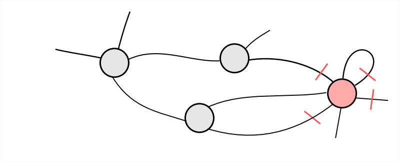
Theorem (Payne-Willwacher '24)

There is an injection

$$H_*(\mathrm{MGC}(g, n, 11)) \hookrightarrow H^*(\mathcal{M}_{g,n})$$

Why 11?

Why MGC(g, n, r)?



Theorem (Payne-Willwacher '24)

There is an injection

$$H_*(\mathrm{MGC}(g, n, 11)) \hookrightarrow H^*(\mathcal{M}_{g,n})$$

Why 11? Reflects $H^i(\overline{\mathcal{M}}_{g,n}) = 0$ for $i = 3, 5, 7, 9$.

Case $g = 1$.

Lemma (W.)

$H_*(\mathrm{MGC}(1, n, k))$ is concentrated in degree $n - k$

Case $g = 1$.

Lemma (W.)

$H_*(\mathrm{MGC}(1, n, k))$ is concentrated in degree $n - k$ with Betti number $s_{n-1,k}$,

Case $g = 1$.

Lemma (W.)

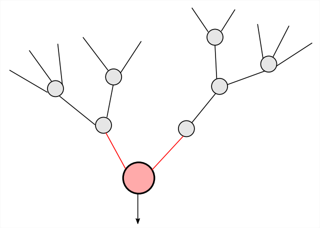
$H_*(\mathrm{MGC}(1, n, k))$ is concentrated in degree $n - k$ with Betti number $s_{n-1,k}$, the number of permutations of $n - 1$ letters with k cycles.

Case $g = 1$.

Lemma (W.)

$H_*(\mathrm{MGC}(1, n, k))$ is concentrated in degree $n - k$ with Betti number $s_{n-1,k}$, the number of permutations of $n - 1$ letters with k cycles.

Proof by picture:

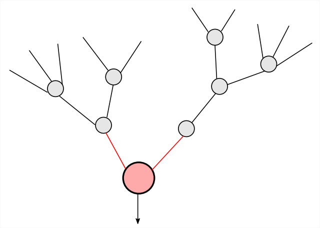


Case $g = 1$.

Lemma (W.)

$H_*(\mathrm{MGC}(1, n, k))$ is concentrated in degree $n - k$ with Betti number $s_{n-1,k}$, the number of permutations of $n - 1$ letters with k cycles.

Proof by picture:



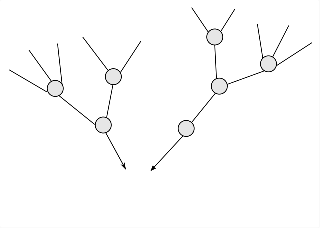
1. Break the symmetry – fix n and only consider the action of S_{n-1}.

Case $g = 1$.

Lemma (W.)

$H_*(\mathrm{MGC}(1, n, k))$ is concentrated in degree $n - k$ with Betti number $s_{n-1,k}$, the number of permutations of $n - 1$ letters with k cycles.

Proof by picture:



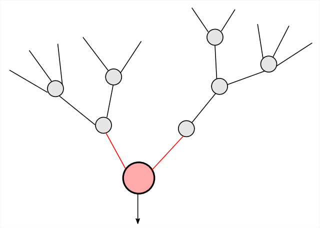
1. Break the symmetry – fix n and only consider the action of S_{n-1}.
2. Reduce to a quotient complex with v on the "bottom".

Case $g = 1$.

Lemma (W.)

$H_*(\mathrm{MGC}(1, n, k))$ is concentrated in degree $n - k$ with Betti number $s_{n-1,k}$, the number of permutations of $n - 1$ letters with k cycles.

Proof by picture:



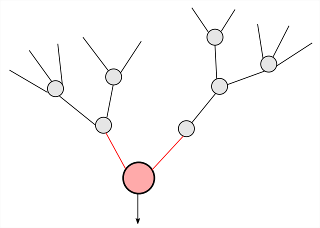
1. Break the symmetry – fix n and only consider the action of S_{n-1}.
2. Reduce to a quotient complex with v on the "bottom".
3. Use knowledge of $H(\mathrm{GC}_{0,n_i})$ to identify the homology of each branch.

Case $g = 1$.

Lemma (W.)

$H_*(\mathrm{MGC}(1, n, k))$ is concentrated in degree $n - k$ with Betti number $s_{n-1,k}$, the number of permutations of $n - 1$ letters with k cycles.

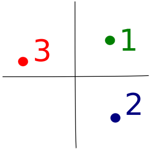
Proof by picture:



1. Break the symmetry – fix n and only consider the action of S_{n-1}.
2. Reduce to a quotient complex with v on the "bottom".
3. Use knowledge of $H(\mathrm{GC}_{0,n_i})$ to identify the homology of each branch.
4. Observe $(n_i - 1)!$ is both the homology of a branch and the number of cycles you can build from n_i letters.

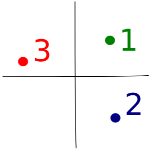
Relation to Configuration Space

Let $C(n, \mathbb{R}^k)$ be the configuration space of n ordered points in $\mathbb{R}^k$.



Relation to Configuration Space

Let $C(n, \mathbb{R}^k)$ be the configuration space of n ordered points in $\mathbb{R}^k$.

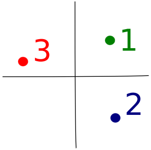


Betti numbers are known, and can be computed recursively. E.g.:

$$dim(H^1(C(n, \mathbb{R}^2))) = \binom{n}{2} = s_{n,n-1}$$

Relation to Configuration Space

Let $C(n, \mathbb{R}^k)$ be the configuration space of n ordered points in $\mathbb{R}^k$.

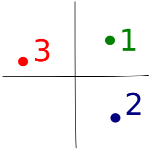


Betti numbers are known, and can be computed recursively. E.g.:

$$dim(H^1(C(n, \mathbb{R}^2))) = \binom{n}{2} = s_{n,n-1}$$

Relation to Configuration Space

Let $C(n, \mathbb{R}^k)$ be the configuration space of n ordered points in $\mathbb{R}^k$.



Betti numbers are known, and can be computed recursively. E.g.:

$$dim(H^1(C(n, \mathbb{R}^2))) = \binom{n}{2} = s_{n,n-1}$$

Theorem (W)

There is an isomorphism of S_{n+1}-representations:

$$H_*(\mathrm{MGC}(1, n+1, k)) \cong H^{2(n-k)}(C(n, \mathbb{R}^3))$$

Detour – Representation Stability

Can consider the rational cohomology as an S_n-representation.

$$H^1(C(n, \mathbb{R}^2)) \cong$$

Detour – Representation Stability

Can consider the rational cohomology as an S_n-representation.

$$H^1(C(n,\mathbb{R}^2)) \cong V_n + V_{n-1,1} + V_{n-2,2}$$

Detour – Representation Stability

Can consider the rational cohomology as an S_n-representation.

$$H^1(C(n,\mathbb{R}^2)) \cong V_n + V_{n-1,1} + V_{n-2,2} = V(0) + V(1) + V(2)$$

Detour – Representation Stability

Can consider the rational cohomology as an S_n-representation.

$$H^1(C(n,\mathbb{R}^2)) \cong V_n + V_{n-1,1} + V_{n-2,2} = V(0) + V(1) + V(2)$$

for $n \geq 4$.

Detour – Representation Stability

Can consider the rational cohomology as an S_n-representation.

$$H^1(C(n, \mathbb{R}^2)) \cong V_n + V_{n-1,1} + V_{n-2,2} = V(0) + V(1) + V(2)$$

for $n \geq 4$.

Theorem (Church & Farb '13)

Fix i and k. The sequence of S_n representations $H^i(C(n, \mathbb{R}^k))$ exhibit representation stability.

Detour – Representation Stability

Can consider the rational cohomology as an S_n-representation.

$$H^1(C(n,\mathbb{R}^2)) \cong V_n + V_{n-1,1} + V_{n-2,2} = V(0) + V(1) + V(2)$$

for $n \geq 4$.

Theorem (Church & Farb '13)

Fix i and k. The sequence of S_n representations $H^i(C(n,\mathbb{R}^k))$ exhibit representation stability.

The same thing happens for $\mathrm{MGC}(g, n+j, k+j)$:

Detour – Representation Stability

Can consider the rational cohomology as an S_n-representation.

$$H^1(C(n,\mathbb{R}^2)) \cong V_n + V_{n-1,1} + V_{n-2,2} = V(0) + V(1) + V(2)$$

for $n \geq 4$.

Theorem (Church & Farb '13)

Fix i and k. The sequence of S_n representations $H^i(C(n,\mathbb{R}^k))$ exhibit representation stability.

The same thing happens for $\mathrm{MGC}(g, n+j, k+j)$:

Detour – Representation Stability

Can consider the rational cohomology as an S_n-representation.

$$H^1(C(n,\mathbb{R}^2)) \cong V_n + V_{n-1,1} + V_{n-2,2} = V(0) + V(1) + V(2)$$

for $n \geq 4$.

Theorem (Church & Farb '13)

Fix i and k. The sequence of S_n representations $H^i(C(n,\mathbb{R}^k))$ exhibit representation stability.

The same thing happens for $\mathrm{MGC}(g, n+j, k+j)$:

Theorem (Fedah-W.)

For each g and $n-k$ the family $\mathrm{MGC}(g, n+j, k+j)$ is representation stable,

Detour – Representation Stability

Can consider the rational cohomology as an S_n-representation.

$$H^1(C(n,\mathbb{R}^2)) \cong V_n + V_{n-1,1} + V_{n-2,2} = V(0) + V(1) + V(2)$$

for $n \geq 4$.

Theorem (Church & Farb '13)

Fix i and k. The sequence of S_n representations $H^i(C(n,\mathbb{R}^k))$ exhibit representation stability.

The same thing happens for $\mathrm{MGC}(g, n+j, k+j)$:

Theorem (Fedah-W.)

For each g and $n-k$ the family $\mathrm{MGC}(g, n+j, k+j)$ is representation stable, with a precisely determined sharp bound.

Detour – Representation Stability

Can consider the rational cohomology as an S_n-representation.

$$H^1(C(n,\mathbb{R}^2)) \cong V_n + V_{n-1,1} + V_{n-2,2} = V(0) + V(1) + V(2)$$

for $n \geq 4$.

Theorem (Church & Farb '13)

Fix i and k. The sequence of S_n representations $H^i(C(n,\mathbb{R}^k))$ exhibit representation stability.

The same thing happens for $\mathrm{MGC}(g, n+j, k+j)$:

Theorem (Fedah-W.)

For each g and $n-k$ the family $\mathrm{MGC}(g, n+j, k+j)$ is representation stable, with a precisely determined sharp bound.

This can be used to establish non-triviality of certain homology classes using Payne and Willwacher's result.

Relation between GC and MGC

GC and MGC aren't just parallel case studies...

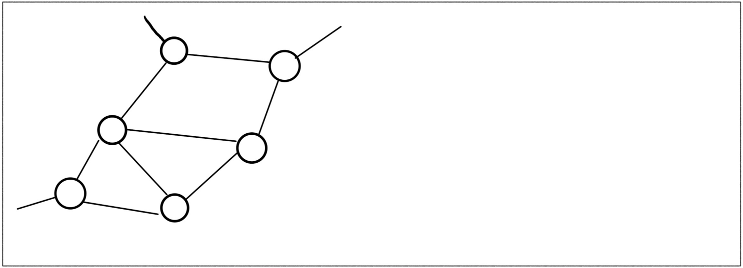
Relation between GC and MGC

GC and MGC aren't just parallel case studies... they're closely related.

Relation between GC and MGC

GC and MGC aren't just parallel case studies... they're closely related.

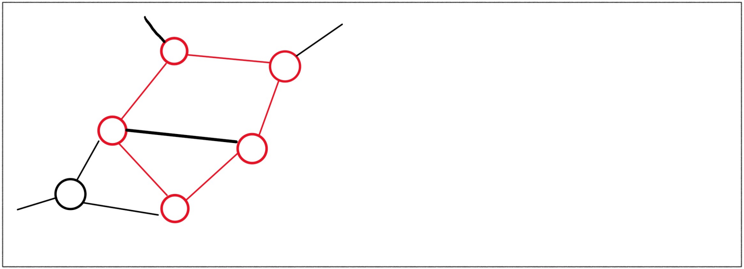
Rough idea: there is a map given by contracting cycles.



Relation between GC and MGC

GC and MGC aren't just parallel case studies... they're closely related.

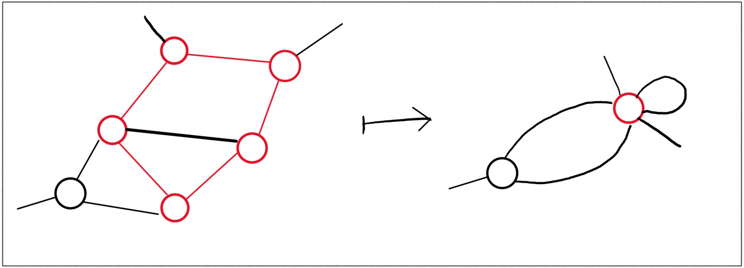
Rough idea: there is a map given by contracting cycles.



Relation between GC and MGC

GC and MGC aren't just parallel case studies... they're closely related.

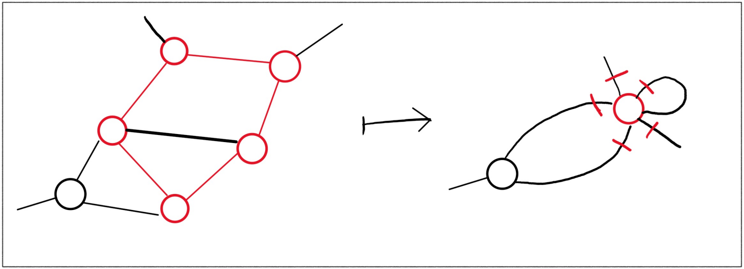
Rough idea: there is a map given by contracting cycles.



Relation between GC and MGC

GC and MGC aren't just parallel case studies... they're closely related.

Rough idea: there is a map given by contracting cycles.

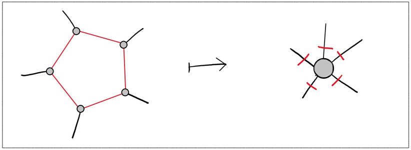


Relation between GC and MGC

Theorem (W)

Cycle contraction gives an isomorphism of graded S_n-modules:

$$H(\mathrm{GC}_{1,n}) \cong \bigoplus_{r \text{ odd}} H(\mathrm{MGC}(1,n,r)).$$

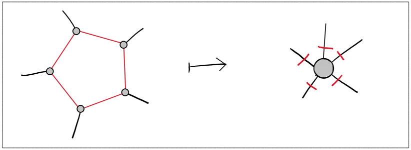


Relation between GC and MGC

Theorem (W)

Cycle contraction gives an isomorphism of graded S_n-modules:

$$H(\mathrm{GC}_{1,n}) \cong \bigoplus_{r \text{ odd}} H(\mathrm{MGC}(1,n,r)).$$



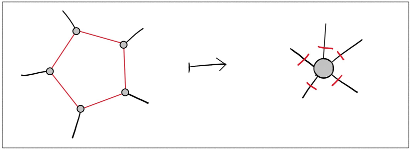
Why Odd?

Relation between GC and MGC

Theorem (W)

Cycle contraction gives an isomorphism of graded S_n-modules:

$$H(\mathrm{GC}_{1,n}) \cong \bigoplus_{r \text{ odd}} H(\mathrm{MGC}(1,n,r)).$$



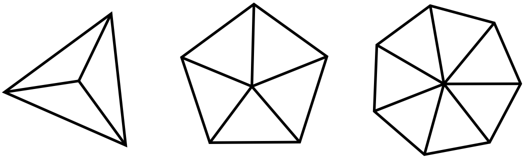
Why Odd?

Corollary

$$\mathrm{gr}_0 H^c_\bullet(\mathcal{M}_{1,n+1}) \cong \bigoplus_i H_{4i}(C(n,\mathbb{R}^3))$$

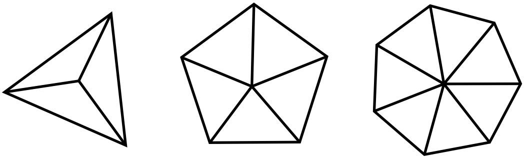
Foreshadowing...

Recall Willwacher used the correspondence with $\mathfrak{grt}_1$ to construct a family of commutative graph homology classes σ_{2j+1}

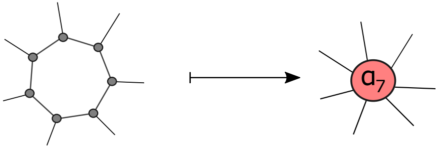


Foreshadowing...

Recall Willwacher used the correspondence with $\mathfrak{grt}_1$ to construct a family of commutative graph homology classes σ_{2j+1}

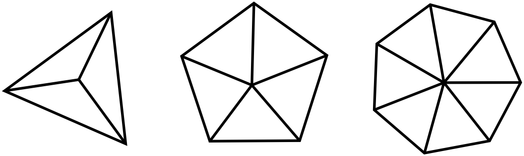


Compare this to the above isomorphism which involved contraction of odd polygons...

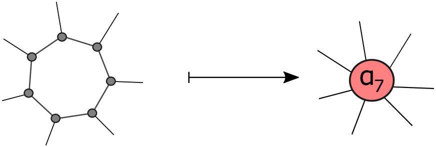


Foreshadowing...

Recall Willwacher used the correspondence with $\mathfrak{grt}_1$ to construct a family of commutative graph homology classes σ_{2j+1}



Compare this to the above isomorphism which involved contraction of odd polygons...



How could we use this to detect the wheel graph in $\mathsf{L}(2j+1,0)$?

Looking ahead...

The above story when $g = 1$ can be generalized:

Looking ahead...

The above story when $g = 1$ can be generalized:

Theorem

There is a short exact sequence of the form:

$$0 \to \mathsf{Ker}(g, n) \hookrightarrow \begin{smallmatrix} \textit{acyclic} \\ \textit{complex} \end{smallmatrix} \twoheadrightarrow \mathsf{GC}_{g,n} \to 0$$

What is this acyclic complex?

Looking ahead...

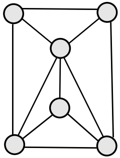
The above story when $g = 1$ can be generalized:

Theorem

There is a short exact sequence of the form:

$$0 \to \mathrm{Ker}(g,n) \hookrightarrow \substack{\textit{acyclic}\\ \textit{complex}} \twoheadrightarrow \mathrm{GC}_{g,n} \to 0$$

What is this acyclic complex?



Looking ahead...

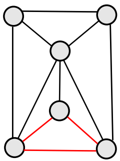
The above story when $g = 1$ can be generalized:

Theorem

There is a short exact sequence of the form:

$$0 \to \mathrm{Ker}(g,n) \hookrightarrow \begin{matrix}\textit{acyclic}\\ \textit{complex}\end{matrix} \twoheadrightarrow \mathsf{GC}_{g,n} \to 0$$

What is this acyclic complex?



Looking ahead...

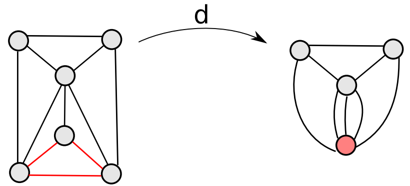
The above story when $g = 1$ can be generalized:

Theorem

There is a short exact sequence of the form:

$$0 \to \mathrm{Ker}(g,n) \hookrightarrow \begin{matrix}\textit{acyclic}\\ \textit{complex}\end{matrix} \twoheadrightarrow \mathrm{GC}_{g,n} \to 0$$

What is this acyclic complex?



Looking ahead...

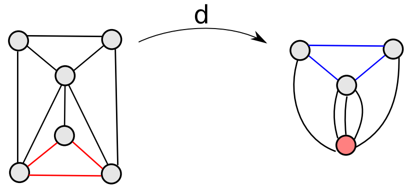
The above story when $g = 1$ can be generalized:

Theorem

There is a short exact sequence of the form:

$$0 \to \mathrm{Ker}(g, n) \hookrightarrow \begin{array}{c}\textit{acyclic}\\ \textit{complex}\end{array} \twoheadrightarrow \mathrm{GC}_{g,n} \to 0$$

What is this acyclic complex?



Looking ahead...

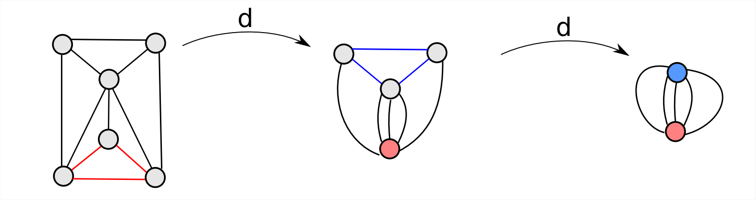
The above story when $g = 1$ can be generalized:

Theorem

There is a short exact sequence of the form:

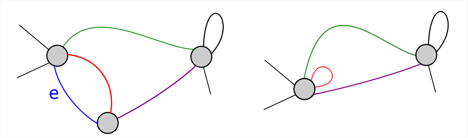
$$0 \to \mathrm{Ker}(g,n) \hookrightarrow \substack{\textit{acyclic} \\ \textit{complex}} \twoheadrightarrow \mathrm{GC}_{g,n} \to 0$$

What is this acyclic complex?

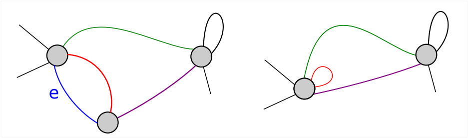


$d = \sum$ contraction of subgraphs.

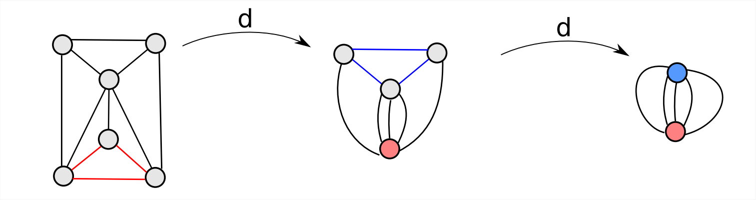
Started with edge contraction:



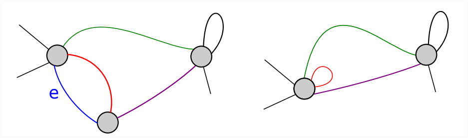
Started with edge contraction:



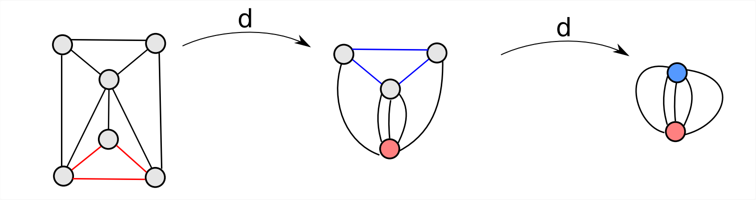
Would like to introduce “higher operations”:

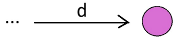


Started with edge contraction:

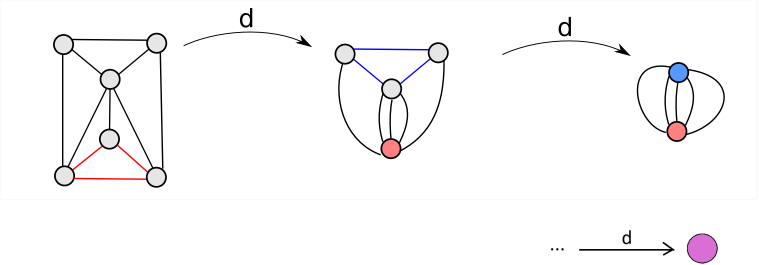


Would like to introduce "higher operations":

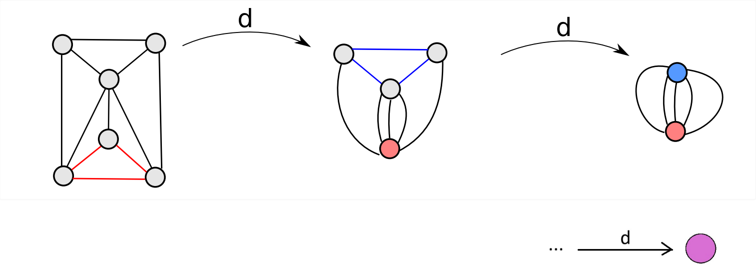




Next time...

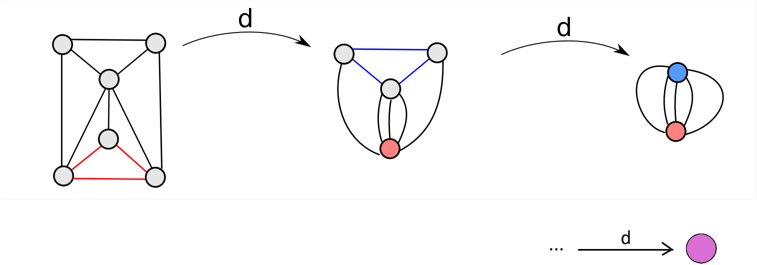


Next time...



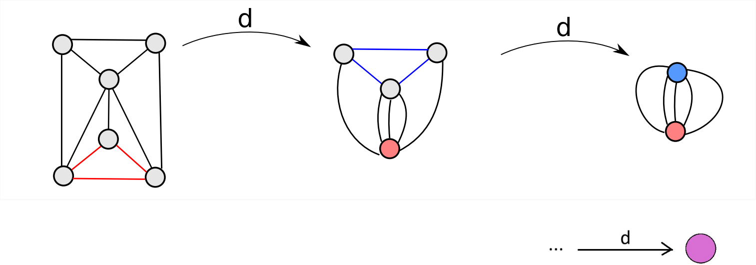
- What information do I need to retain when contracting subgraphs.

Next time...



- What information do I need to retain when contracting subgraphs.
- Why would variations on the graph complex construction be related?

Next time...



- What information do I need to retain when contracting subgraphs.
- Why would variations on the graph complex construction be related?

Higher operations arise from an analogy

Associative Algebras :: Modular operads