Introduction to Graph Complexes – II

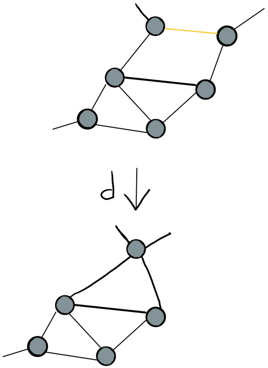
Ben Ward

Bowling Green State University

IISER – Kolkata
November 2025

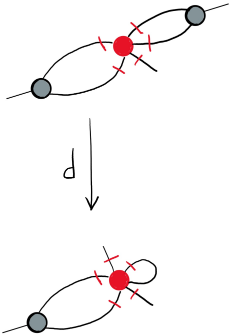
Last Time...

We looked at a chain complex GC built from graphs.



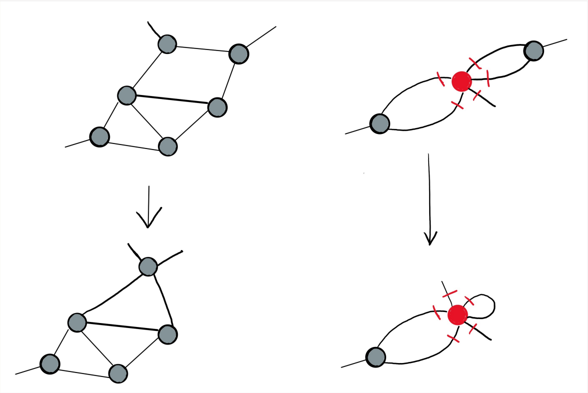
Last Time...

Then we looked at a chain complex built from decorated graphs MGC.



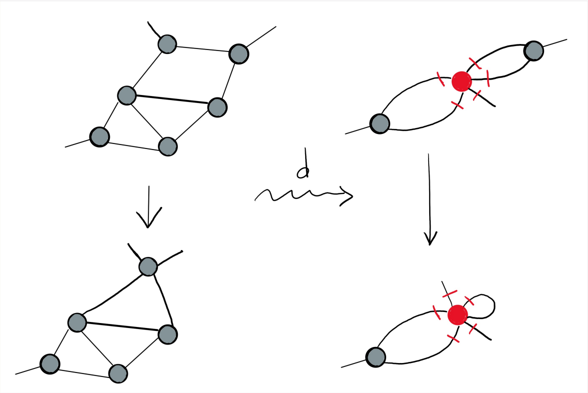
Last Time...

These chain complex each compute homology of interesting spaces.



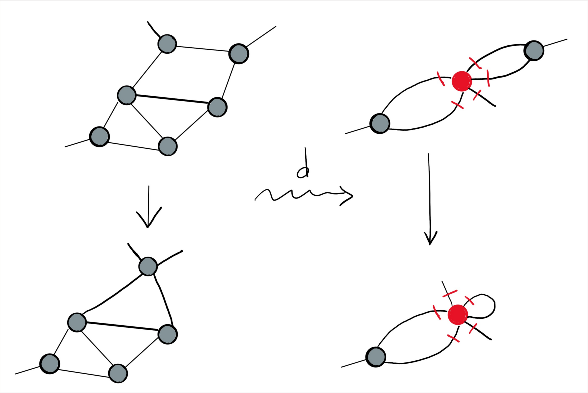
Last Time...

Today: indicate how they're related...



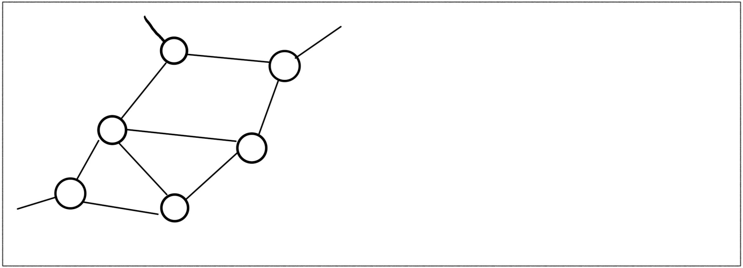
Last Time...

Today: indicate how they're related via higher structures.



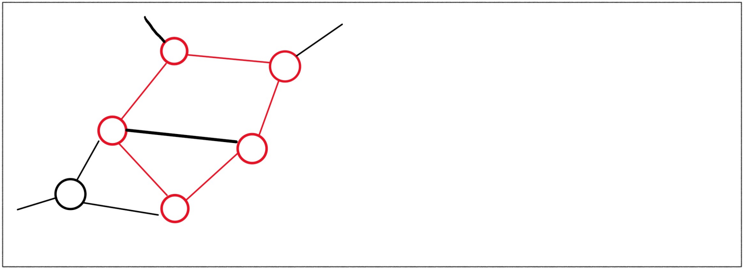
Relation between GC and MGC

Rough idea: there is a map given by contracting cycles.



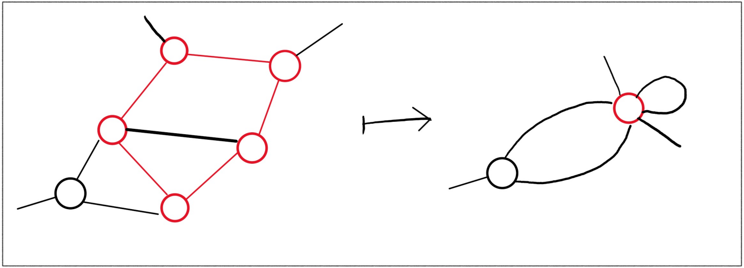
Relation between GC and MGC

Rough idea: there is a map given by contracting cycles.



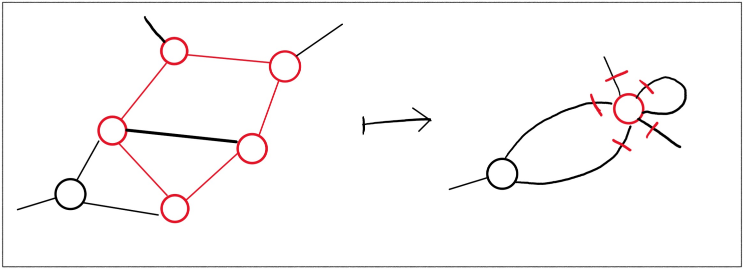
Relation between GC and MGC

Rough idea: there is a map given by contracting cycles.



Relation between GC and MGC

Rough idea: there is a map given by contracting cycles.

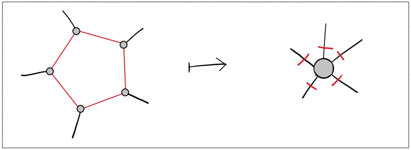


Relation between GC and MGC

Theorem (W)

Cycle contraction gives an isomorphism of graded S_n-modules:

$$H(\mathrm{GC}_{1,n}) \cong \bigoplus_{r \text{ odd}} H(\mathrm{MGC}(1,n,r)).$$

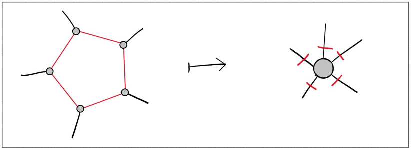


Relation between GC and MGC

Theorem (W)

Cycle contraction gives an isomorphism of graded S_n-modules:

$$H(\mathrm{GC}_{1,n}) \cong \bigoplus_{r \text{ odd}} H(\mathrm{MGC}(1, n, r)).$$



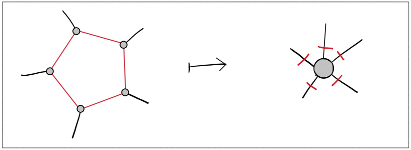
Why Odd?

Relation between GC and MGC

Theorem (W)

Cycle contraction gives an isomorphism of graded S_n-modules:

$$H(\mathrm{GC}_{1,n}) \cong \bigoplus_{r \text{ odd}} H(\mathrm{MGC}(1,n,r)).$$



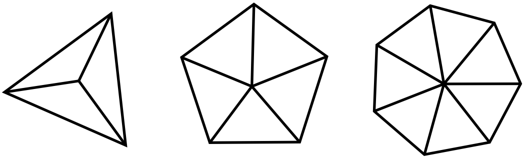
Why Odd?

Corollary

$$\mathrm{gr}_0 H^c_\bullet(\mathcal{M}_{1,n+1}) \cong \bigoplus_i H_{4i}(C(n,\mathbb{R}^3))$$

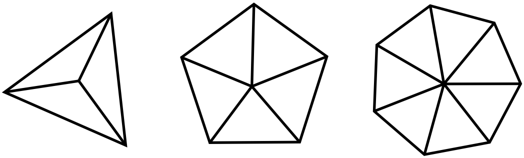
Foreshadowing...

Recall Willwacher used the correspondence with $\mathfrak{grt}_1$ to construct a family of commutative graph homology classes σ_{2j+1}

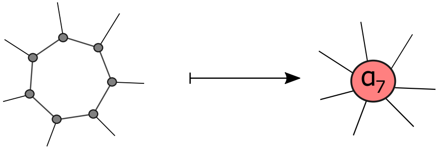


Foreshadowing...

Recall Willwacher used the correspondence with $\mathfrak{grt}_1$ to construct a family of commutative graph homology classes σ_{2j+1}

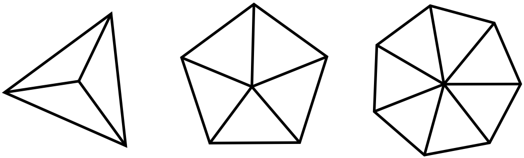


Compare this to the above isomorphism which involved contraction of odd polygons...

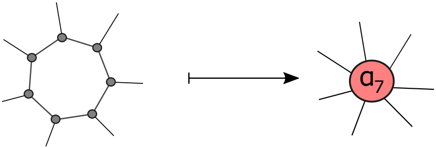


Foreshadowing...

Recall Willwacher used the correspondence with $\mathfrak{grt}_1$ to construct a family of commutative graph homology classes σ_{2j+1}



Compare this to the above isomorphism which involved contraction of odd polygons...



How could we use this to detect the wheel graph in $\mathsf{L}(2j+1, 0)$?

Look at $g > 1$ case

The $g = 1$ story can be generalized:

Look at $g > 1$ case

The $g = 1$ story can be generalized:

Theorem

There is a short exact sequence of the form:

$$0 \to \mathrm{Ker}(g,n) \hookrightarrow \begin{smallmatrix}\text{acyclic}\\ \text{complex}\end{smallmatrix} \twoheadrightarrow \mathrm{GC}_{g,n} \to 0$$

What is this acyclic complex?

Look at $g > 1$ case

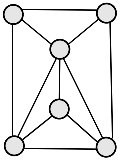
The $g = 1$ story can be generalized:

Theorem

There is a short exact sequence of the form:

$$0 \to \mathrm{Ker}(g,n) \hookrightarrow \substack{\textit{acyclic}\\ \textit{complex}} \twoheadrightarrow \mathrm{GC}_{g,n} \to 0$$

What is this acyclic complex?



Look at $g > 1$ case

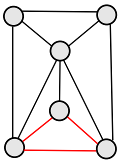
The $g = 1$ story can be generalized:

Theorem

There is a short exact sequence of the form:

$$0 \to \mathrm{Ker}(g,n) \hookrightarrow \begin{matrix} \textit{acyclic} \\ \textit{complex} \end{matrix} \twoheadrightarrow \mathrm{GC}_{g,n} \to 0$$

What is this acyclic complex?



Look at $g > 1$ case

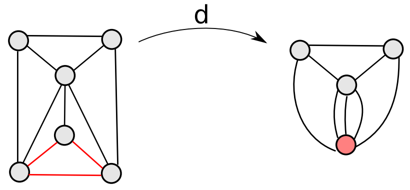
The $g = 1$ story can be generalized:

Theorem

There is a short exact sequence of the form:

$$0 \to \mathrm{Ker}(g, n) \hookrightarrow \begin{matrix} \textit{acyclic} \\ \textit{complex} \end{matrix} \twoheadrightarrow \mathrm{GC}_{g,n} \to 0$$

What is this acyclic complex?



Look at $g > 1$ case

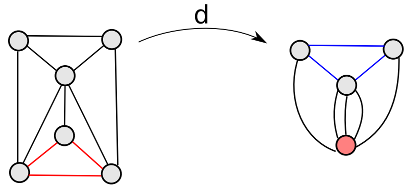
The $g = 1$ story can be generalized:

Theorem

There is a short exact sequence of the form:

$$0 \to \mathrm{Ker}(g, n) \hookrightarrow \begin{matrix}\textit{acyclic}\\ \textit{complex}\end{matrix} \twoheadrightarrow \mathrm{GC}_{g,n} \to 0$$

What is this acyclic complex?



Look at $g > 1$ case

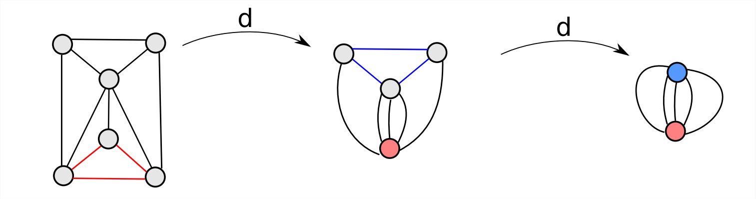
The $g = 1$ story can be generalized:

Theorem

There is a short exact sequence of the form:

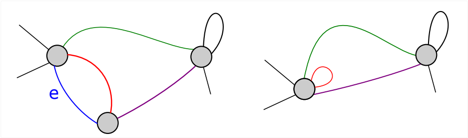
$$0 \to \mathrm{Ker}(g,n) \hookrightarrow \begin{array}{c}\textit{acyclic}\\ \textit{complex}\end{array} \twoheadrightarrow \mathrm{GC}_{g,n} \to 0$$

What is this acyclic complex?

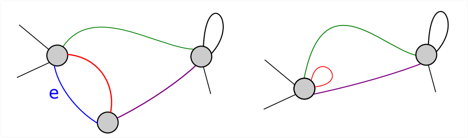


$d = \sum$ contraction of subgraphs.

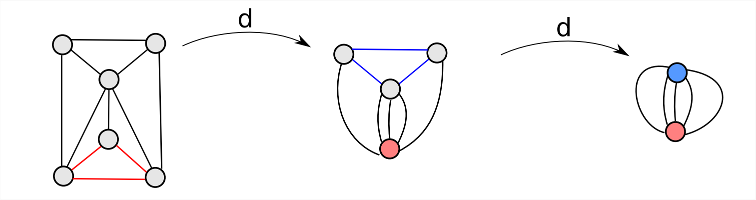
Started with edge contraction:



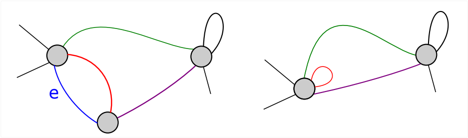
Started with edge contraction:



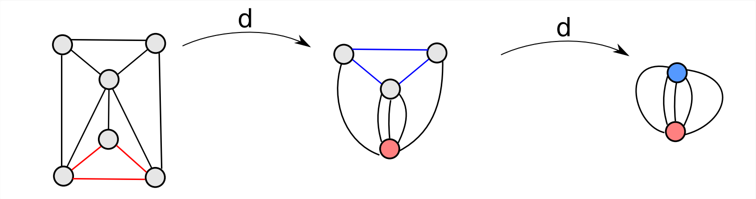
Would like to introduce “higher operations”:

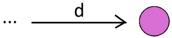


Started with edge contraction:

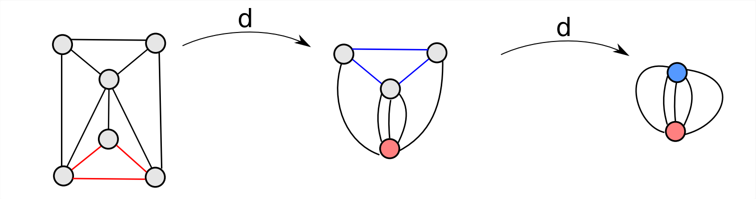


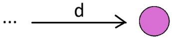
Would like to introduce “higher operations”:



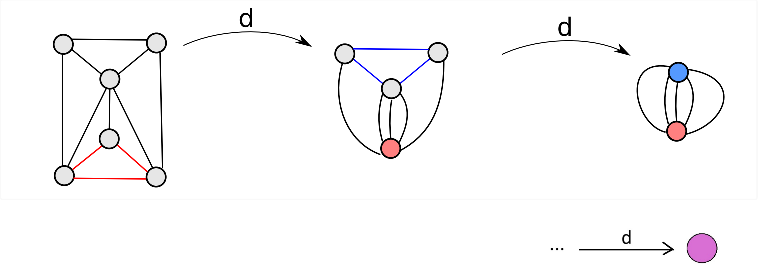


Today's questions



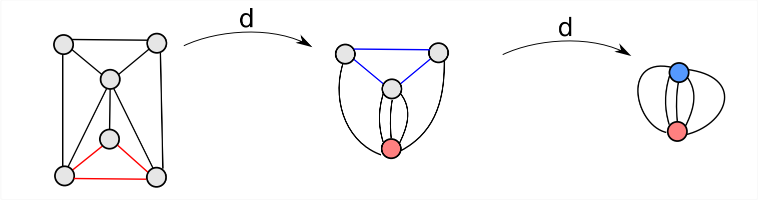


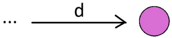
Today's questions



- What information do I need to retain when contracting subgraphs.

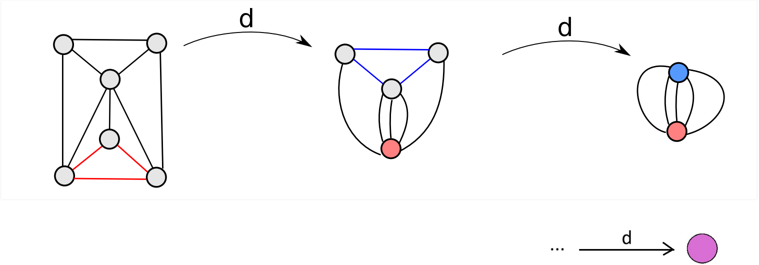
Today's questions





- What information do I need to retain when contracting subgraphs.
- Why would variations on the graph complex construction be related?

Today's questions



- What information do I need to retain when contracting subgraphs.
- Why would variations on the graph complex construction be related?

Higher operations arise from an analogy

Associative Algebras :: Modular operads

Recollection of A_∞-algebras

Definition (Stasheff): The associahedron K_n is a polytope of dimension $n-2$ such that:

Recollection of A_∞-algebras

Definition (Stasheff): The associahedron K_n is a polytope of dimension $n-2$ such that:

- vertices $=$ ways to parenthesize a string of n letters,

Recollection of A_∞-algebras

Definition (Stasheff): The associahedron K_n is a polytope of dimension $n-2$ such that:

- vertices $=$ ways to parenthesize a string of n letters,
- edges $=$ strings of n letters missing one pair of parentheses,

Recollection of A_∞-algebras

Definition (Stasheff): The associahedron K_n is a polytope of dimension $n-2$ such that:

- vertices $=$ ways to parenthesize a string of n letters,
- edges $=$ strings of n letters missing one pair of parentheses,
- faces $=$ strings of n letters missing two pair of parentheses

Recollection of A_∞-algebras

Definition (Stasheff): The associahedron K_n is a polytope of dimension $n-2$ such that:

- vertices $=$ ways to parenthesize a string of n letters,
- edges $=$ strings of n letters missing one pair of parentheses,
- faces $=$ strings of n letters missing two pair of parentheses and so on,
- adjacency is determined by adding parentheses.

Recollection of A_∞-algebras

Definition (Stasheff): The associahedron K_n is a polytope of dimension $n-2$ such that:

- vertices $=$ ways to parenthesize a string of n letters,
- edges $=$ strings of n letters missing one pair of parentheses,
- faces $=$ strings of n letters missing two pair of parentheses and so on,
- adjacency is determined by adding parentheses.

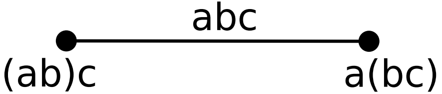
Case $n=3$:

Recollection of A_∞-algebras

Definition (Stasheff): The associahedron K_n is a polytope of dimension $n-2$ such that:

- vertices $=$ ways to parenthesize a string of n letters,
- edges $=$ strings of n letters missing one pair of parentheses,
- faces $=$ strings of n letters missing two pair of parentheses and so on,
- adjacency is determined by adding parentheses.

Case $n = 3$:



Recollection of A_∞-algebras

Example of K_4.

String of
4 letters

Polytope

abcd

Recollection of A_∞-algebras

Example of K_4.

String of 4 letters	Polytope
((ab)c)d	

Recollection of A_∞-algebras

Example of K_4.

String of 4 letters	Polytope
((ab)c)d	•

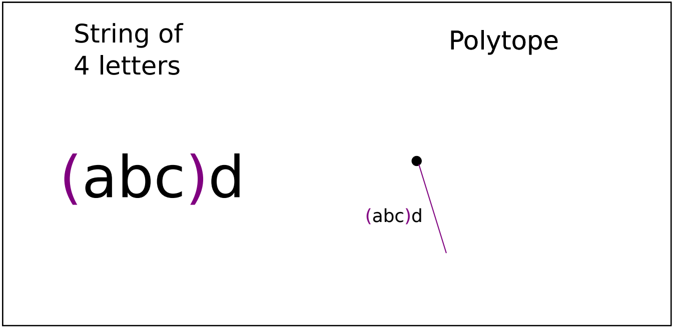
Recollection of A_∞-algebras

Example of K_4.

String of 4 letters	Polytope
$((ab)c)d$	$\bullet$

Recollection of A_∞-algebras

Example of K_4.



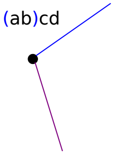
Recollection of A_∞-algebras

Example of K_4.

String of
4 letters

(ab)cd

Polytope



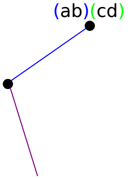
Recollection of A_∞-algebras

Example of K_4.

String of
4 letters

(ab)(cd)

Polytope



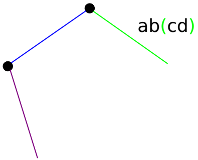
Recollection of A_∞-algebras

Example of K_4.

String of
4 letters

ab(cd)

Polytope



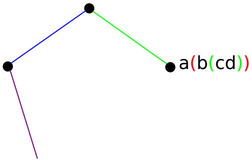
Recollection of A_∞-algebras

Example of K_4.

String of
4 letters

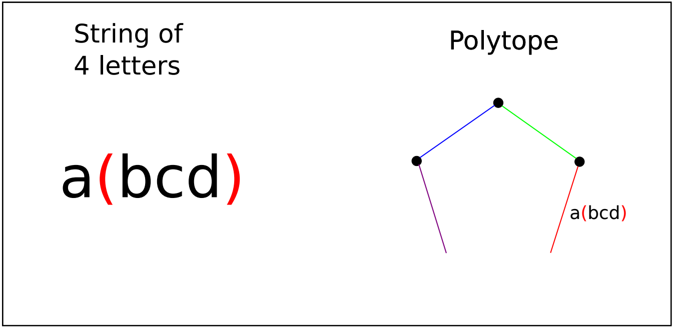
a(b(cd))

Polytope



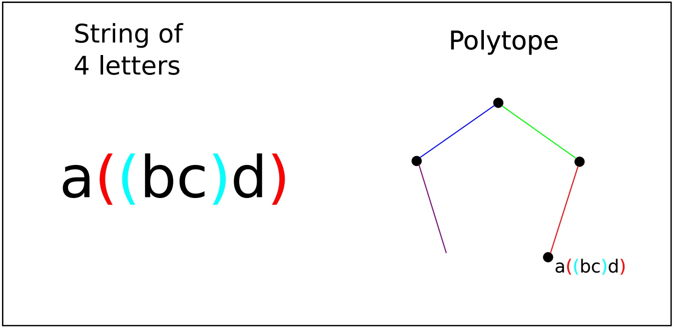
Recollection of A_∞-algebras

Example of K_4.



Recollection of A_∞-algebras

Example of K_4.



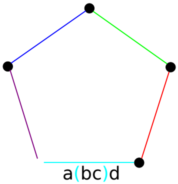
Recollection of A_∞-algebras

Example of K_4.

String of
4 letters

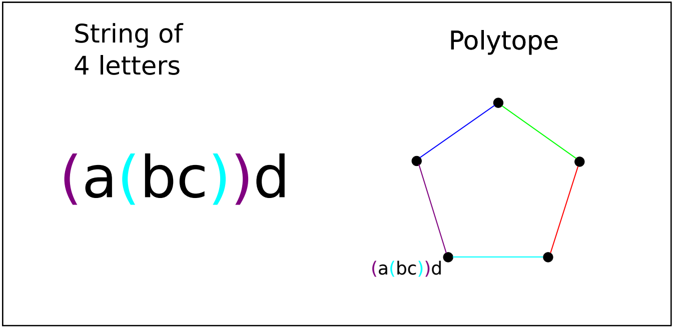
a(bc)d

Polytope



Recollection of A_∞-algebras

Example of K_4.



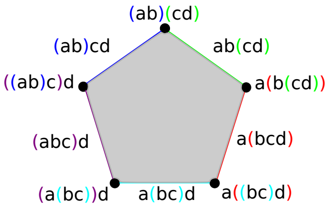
Recollection of A_∞-algebras

Example of K_4.

String of
4 letters

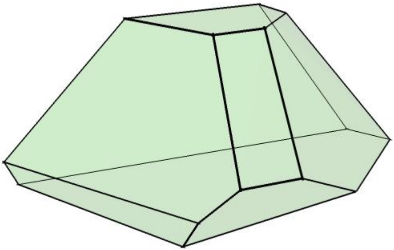
abcd

Polytope



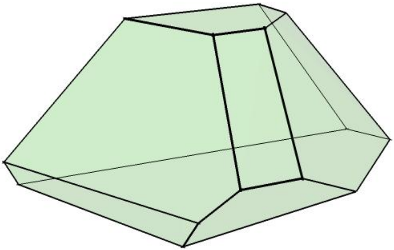
Associahedra

Example: K_5



Associahedra

Example: K_5



Example: $K_2 = \bullet$

Associahedra encode A_∞-algebras

Informal **Definition**:

Associahedra encode A_∞-algebras

Informal **Definition**: An A_∞ algebra is a chain complex A with a compatible linear map $A^{\otimes n} \to A$ for every cell in K_n (for each n).

Associahedra encode A_∞-algebras

Informal **Definition**: An A_∞ algebra is a chain complex A with a compatible linear map $A^{\otimes n} \to A$ for every cell in K_n (for each n).

What is needed to specify an A_∞-algebra?

Associahedra encode A_∞-algebras

Informal **Definition**: An A_∞ algebra is a chain complex A with a compatible linear map $A^{\otimes n} \to A$ for every cell in K_n (for each n).

What is needed to specify an A_∞-algebra?

K_2	ab •	need $\mu_2 \colon A^{\otimes 2} \to A$ of degree 0

Associahedra encode A_∞-algebras

Informal **Definition**: An A_∞ algebra is a chain complex A with a compatible linear map $A^{\otimes n} \to A$ for every cell in K_n (for each n).

What is needed to specify an A_∞-algebra?

K_2		need $\mu_2\colon A^{\otimes 2} \to A$ of degree 0
K_3		

Associahedra encode A_∞-algebras

Informal **Definition**: An A_∞ algebra is a chain complex A with a compatible linear map $A^{\otimes n} \to A$ for every cell in K_n (for each n).

What is needed to specify an A_∞-algebra?

K_2		need μ_2: $A^{\otimes 2} \to A$ of degree 0
K_3		need μ_3 : $A^{\otimes 3} \to A$ of degree 1

Associahedra encode A_∞-algebras

Informal **Definition**: An A_∞ algebra is a chain complex A with a compatible linear map $A^{\otimes n} \to A$ for every cell in K_n (for each n).

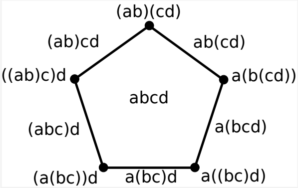
What is needed to specify an A_∞-algebra?

K_2	 ab	need μ_2: $A^{\otimes 2} \to A$ of degree 0
K_3	 (ab)c abc a(bc)	need μ_3 : $A^{\otimes 3} \to A$ of degree 1

Associahedra encode A_∞-algebras

Informal **Definition**: An A_∞ algebra is a chain complex A with a compatible linear map $A^{\otimes n} \to A$ for every cell in K_n (for each n).

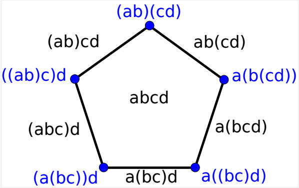
What is needed to specify an A_∞-algebra?

<table>
<tr><td>K_2</td><td></td><td>need $\mu_2 : A^{\otimes 2} \to A$ of degree 0</td></tr>
<tr><td>K_3</td><td></td><td>need $\mu_3 : A^{\otimes 3} \to A$ of degree 1</td></tr>
<tr><td>K_4</td><td></td><td>need $\mu_4 : A^{\otimes 4} \to A$ of degree 2</td></tr>
</table>

Associahedra encode A_∞-algebras

Informal **Definition**: An A_∞ algebra is a chain complex A with a compatible linear map $A^{\otimes n} \to A$ for every cell in K_n (for each n).

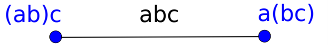
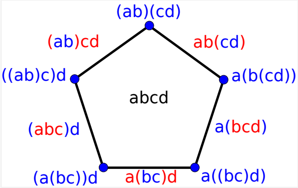
What is needed to specify an A_∞-algebra?

<table>
<tr><td>K_2</td><td>
</td><td>need $\mu_2 : A^{\otimes 2} \to A$ of degree 0</td></tr>
<tr><td>K_3</td><td>
</td><td>need $\mu_3 : A^{\otimes 3} \to A$ of degree 1</td></tr>
<tr><td>K_4</td><td>
</td><td>need $\mu_4 : A^{\otimes 4} \to A$ of degree 2</td></tr>
</table>

Associahedra encode A_∞-algebras

Informal **Definition**: An A_∞ algebra is a chain complex A with a compatible linear map $A^{\otimes n} \to A$ for every cell in K_n (for each n).

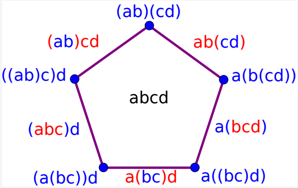
What is needed to specify an A_∞-algebra?

K_2		need $\mu_2 : A^{\otimes 2} \to A$ of degree 0
K_3		need $\mu_3 : A^{\otimes 3} \to A$ of degree 1
K_4		need $\mu_4 : A^{\otimes 4} \to A$ of degree 2

Associahedra encode A_∞-algebras

Informal **Definition**: An A_∞ algebra is a chain complex A with a compatible linear map $A^{\otimes n} \to A$ for every cell in K_n (for each n).

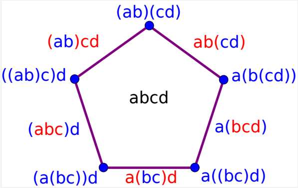
What is needed to specify an A_∞-algebra?

K_2		need $\mu_2 \colon A^{\otimes 2} \to A$ of degree 0
K_3		need $\mu_3 : A^{\otimes 3} \to A$ of degree 1
K_4		need $\mu_4 \colon A^{\otimes 4} \to A$ of degree 2

Associahedra encode A_∞-algebras

Informal **Definition**: An A_∞ algebra is a chain complex A with a compatible linear map $A^{\otimes n} \to A$ for every cell in K_n (for each n).

What is needed to specify an A_∞-algebra?

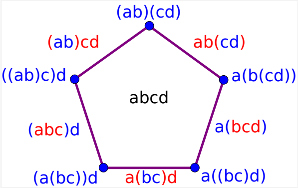
K_2		need $\mu_2\colon A^{\otimes 2} \to A$ of degree 0
K_3		need $\mu_3 : A^{\otimes 3} \to A$ of degree 1
K_4		need $\mu_4\colon A^{\otimes 4} \to A$ of degree 2

and so on...

Associahedra encode A_∞-algebras

Informal **Definition**: An A_∞ algebra is a chain complex A with a compatible linear map $A^{\otimes n} \to A$ for every cell in K_n (for each n).

What is needed to specify an A_∞-algebra?

K_2		need μ_2: $A^{\otimes 2} \to A$ of degree 0
K_3		need μ_3 : $A^{\otimes 3} \to A$ of degree 1
K_4		need μ_4: $A^{\otimes 4} \to A$ of degree 2

and so on... A_∞-algebra $= (A, \mu_2, \mu_3, \mu_4, \mu_5, \ldots)$

Why A_∞ algebras?

Theorem (Kadeishvili)

Let A be a dg associative algebra over a field of characteristic zero. There exists an A_∞ structure on $H_*(A)$ such that $A \sim H_*(A)$ as A_∞-algebras.

Why A_∞ algebras?

Theorem (Kadeishvili)

Let A be a dg associative algebra over a field of characteristic zero. There exists an A_∞ structure on $H_*(A)$ such that $A \sim H_*(A)$ as A_∞-algebras.

For example let $A = C^*(X;\mathbb{Q})$ with the cup product.

Why A_∞ algebras?

Theorem (Kadeishvili)

Let A be a dg associative algebra over a field of characteristic zero. There exists an A_∞ structure on $H_*(A)$ such that $A \sim H_*(A)$ as A_∞-algebras.

For example let $A = C^*(X;\mathbb{Q})$ with the cup product. This is a finer invariant than cohomology, $H^*(X;\mathbb{Q})$

Why A_∞ algebras?

Theorem (Kadeishvili)

Let A be a dg associative algebra over a field of characteristic zero. There exists an A_∞ structure on $H_*(A)$ such that $A \sim H_*(A)$ as A_∞-algebras.

For example let $A = C^*(X;\mathbb{Q})$ with the cup product. This is a finer invariant than cohomology, $H^*(X;\mathbb{Q})$ until we add higher operations

$$\mu_n \colon H^*(A)^{\otimes n} \to H^{*-n+2}(A).$$

Why A_∞ algebras?

Theorem (Kadeishvili)

Let A be a dg associative algebra over a field of characteristic zero. There exists an A_∞ structure on $H_*(A)$ such that $A \sim H_*(A)$ as A_∞-algebras.

For example let $A = C^*(X; \mathbb{Q})$ with the cup product. This is a finer invariant than cohomology, $H^*(X; \mathbb{Q})$ until we add higher operations

$$\mu_n \colon H^*(A)^{\otimes n} \to H^{*-n+2}(A).$$

- For X simply connected, $(H^*(A), \mu_n)$ is a complete invariant of the rational homotopy type.
- We will call these higher operations "Massey products".

Associativity revisited

Key feature of associahedra: they are contractible.

Associativity revisited

Key feature of associahedra: they are contractible.

- I claim this feature is present in much greater generality.

Associativity revisited

Key feature of associahedra: they are contractible.

- I claim this feature is present in much greater generality.

- Revisit associativity:

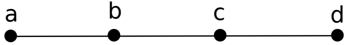
Associativity revisited

Key feature of associahedra: they are contractible.

- I claim this feature is present in much greater generality.

- Revisit associativity:

abcd =

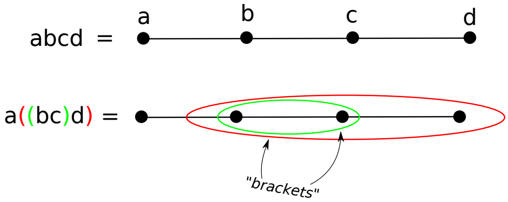


Associativity revisited

Key feature of associahedra: they are contractible.

- I claim this feature is present in much greater generality.

- Revisit associativity:

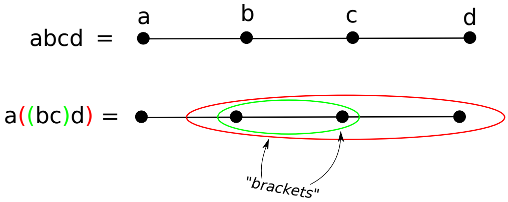


Associativity revisited

Key feature of associahedra: they are contractible.

- I claim this feature is present in much greater generality.

- Revisit associativity:



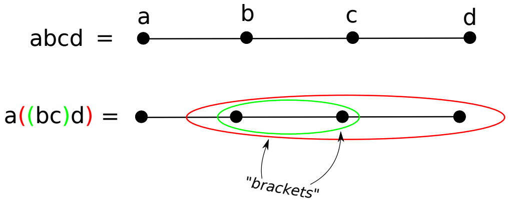
- brackets are either nested or disjoint

Associativity revisited

Key feature of associahedra: they are contractible.

- I claim this feature is present in much greater generality.

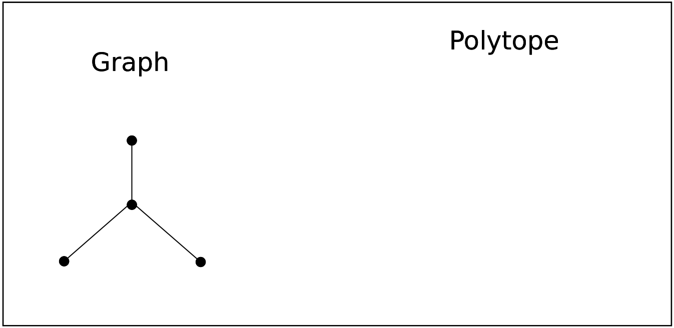
- Revisit associativity:



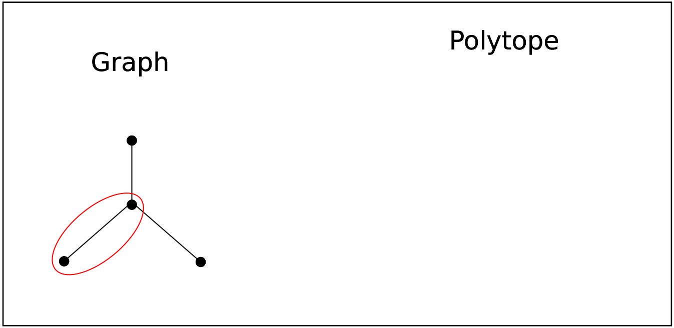
  - brackets are either nested or disjoint

Lemma (W.) The space of bracketings of *any graph* is contractible, in fact it is a polytope.

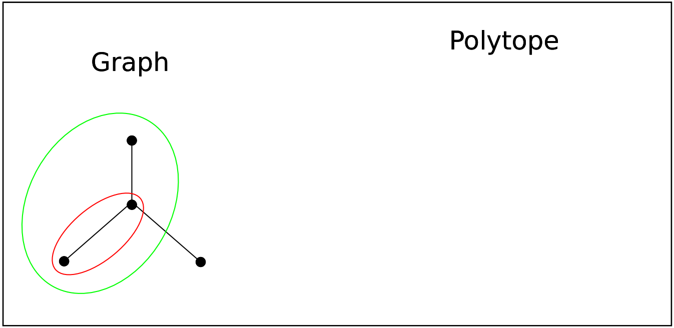
The space of bracketings of any graph is contractible



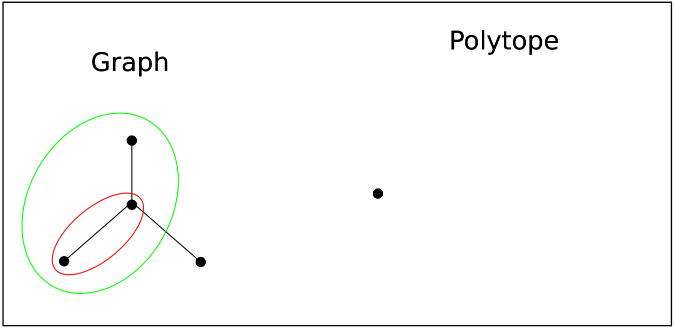
The space of bracketings of any graph is contractible



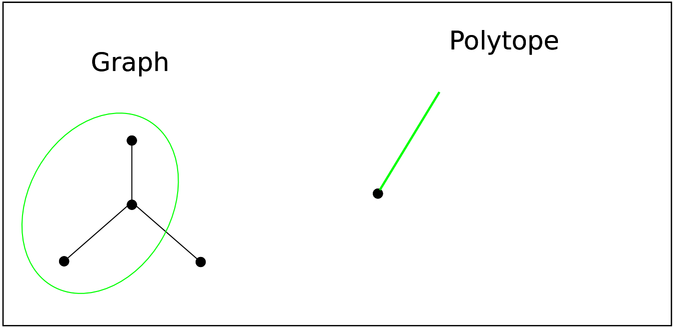
The space of bracketings of any graph is contractible



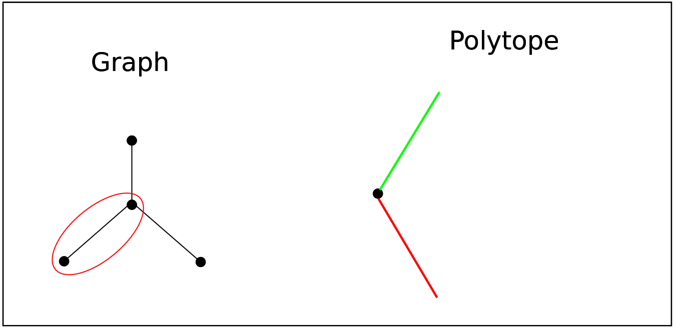
The space of bracketings of any graph is contractible



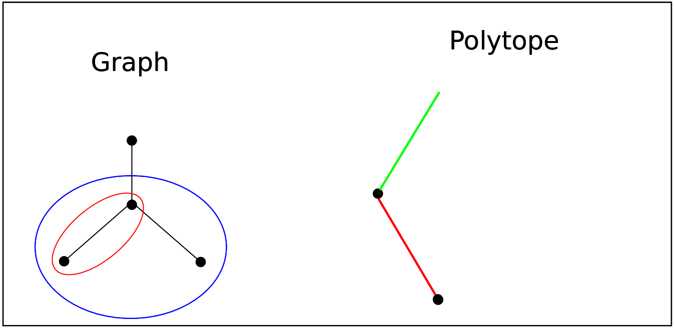
The space of bracketings of any graph is contractible



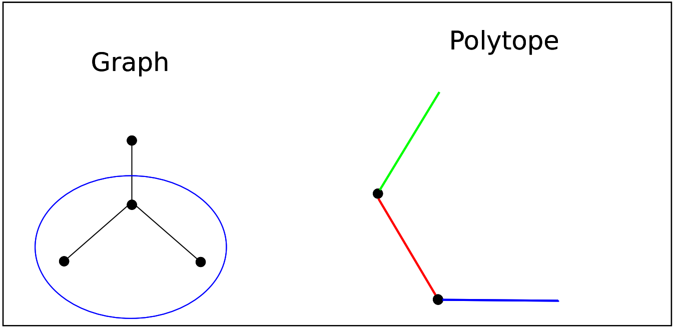
The space of bracketings of any graph is contractible



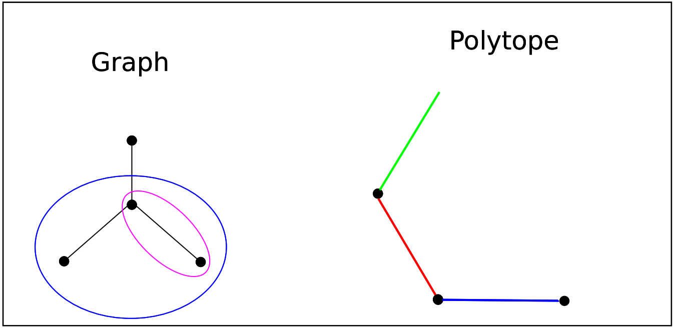
The space of bracketings of any graph is contractible



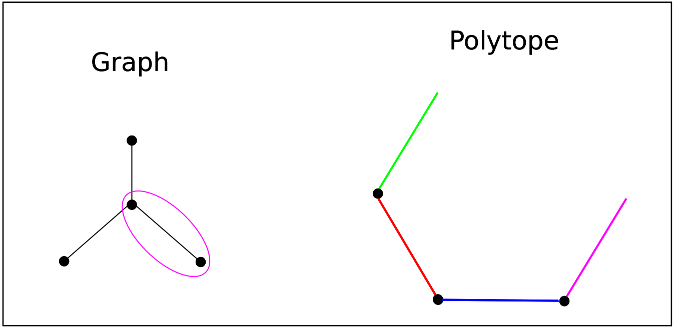
The space of bracketings of any graph is contractible



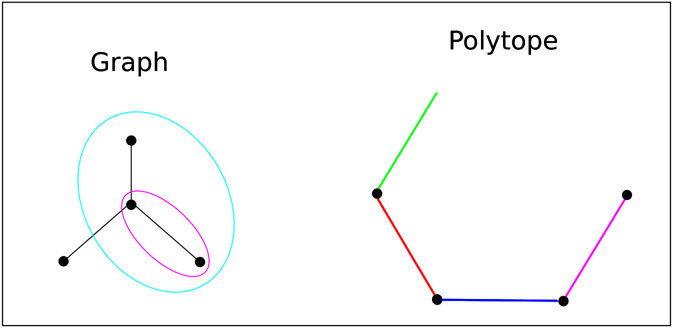
The space of bracketings of any graph is contractible



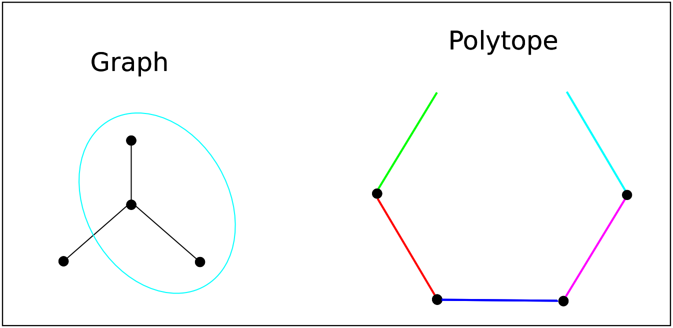
The space of bracketings of any graph is contractible



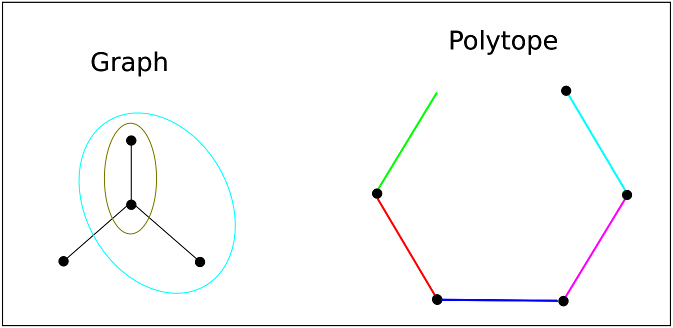
The space of bracketings of any graph is contractible



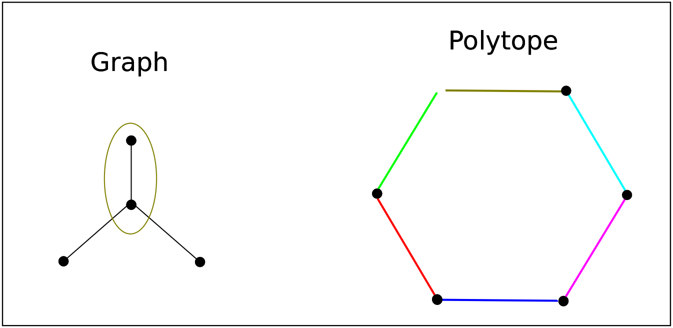
The space of bracketings of any graph is contractible



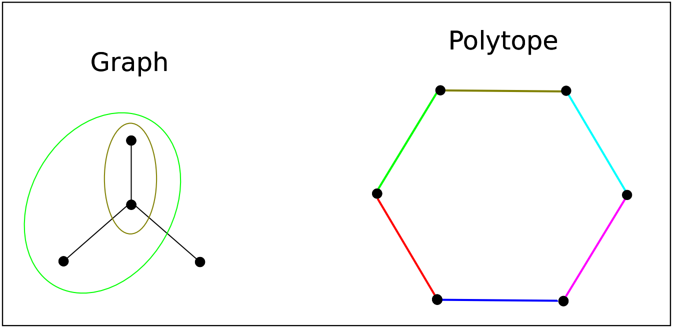
The space of bracketings of any graph is contractible



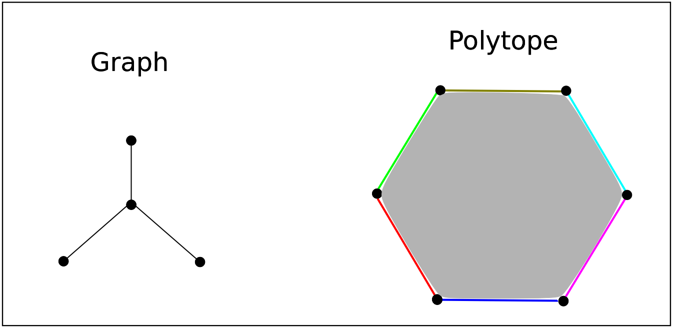
The space of bracketings of any graph is contractible



The space of bracketings of any graph is contractible

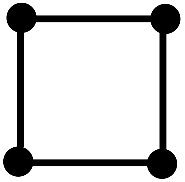


The space of bracketings of any graph is contractible



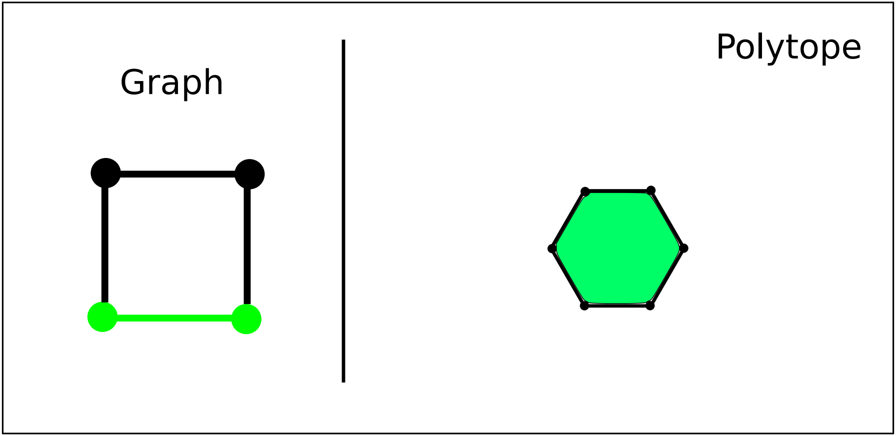
Example of a square graph.

Graph

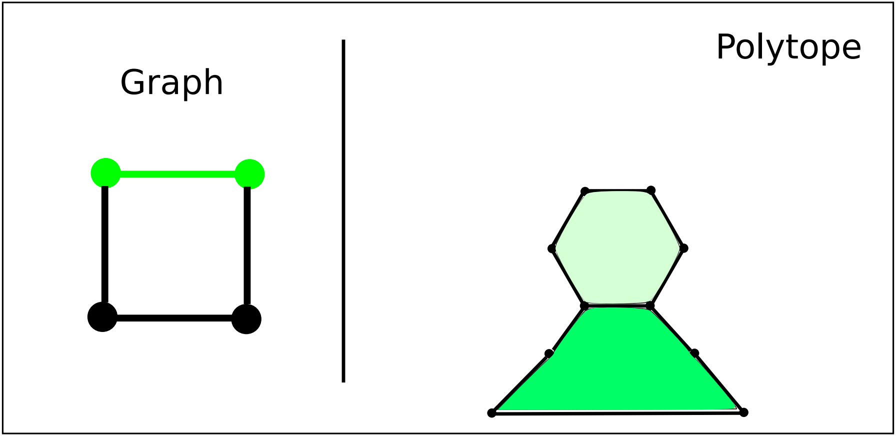


Polytope

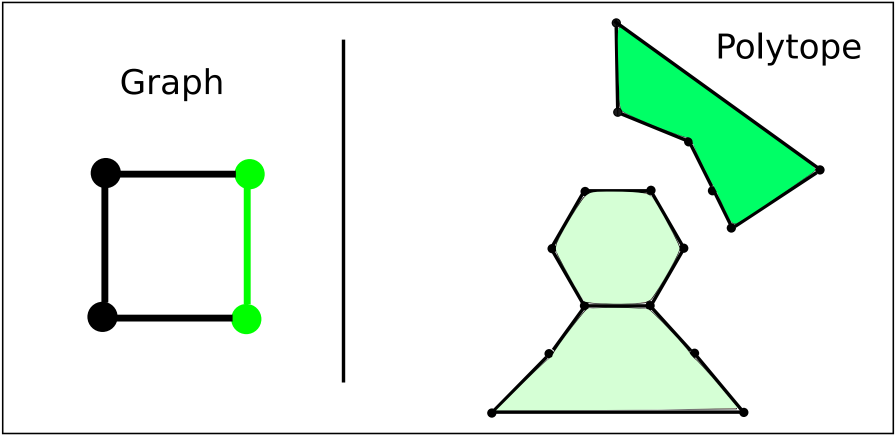
Example of a square graph.



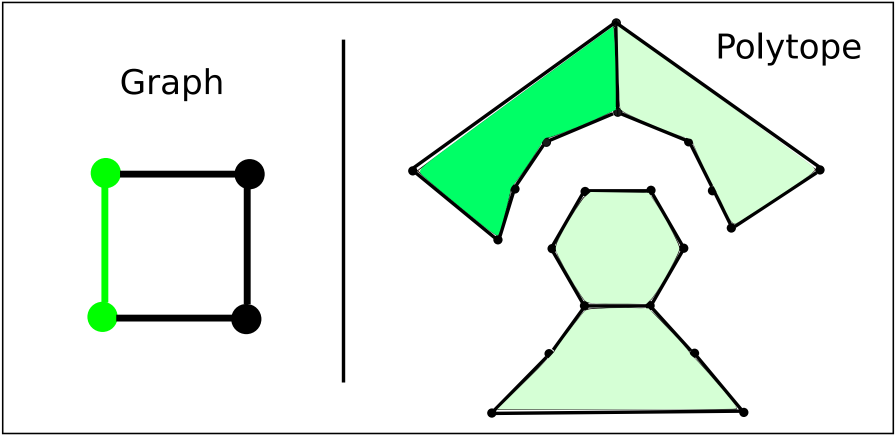
Example of a square graph.



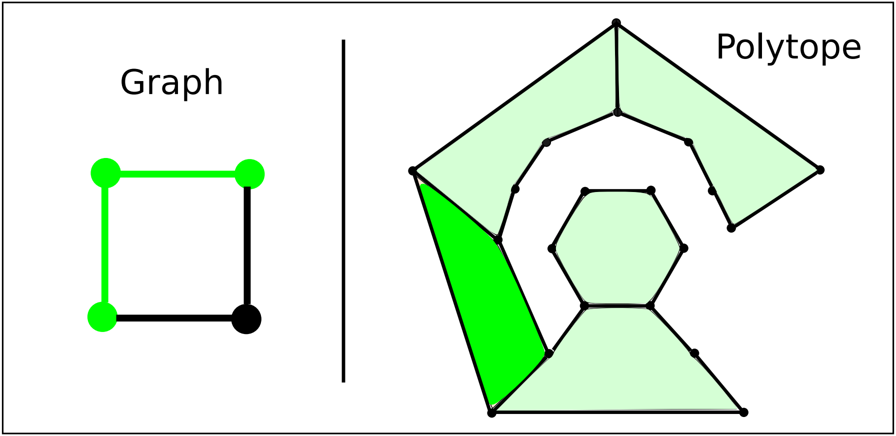
Example of a square graph.



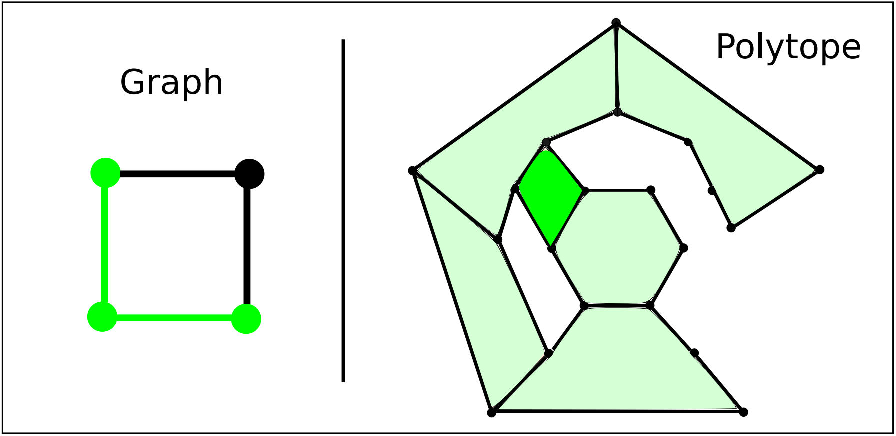
Example of a square graph.



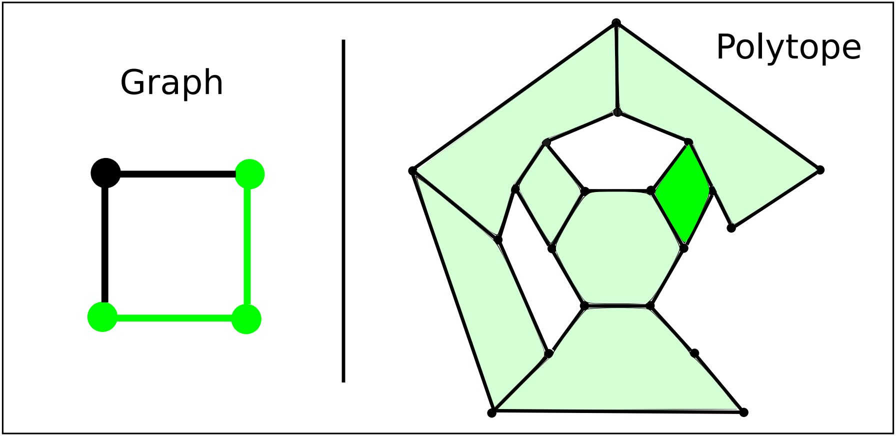
Example of a square graph.



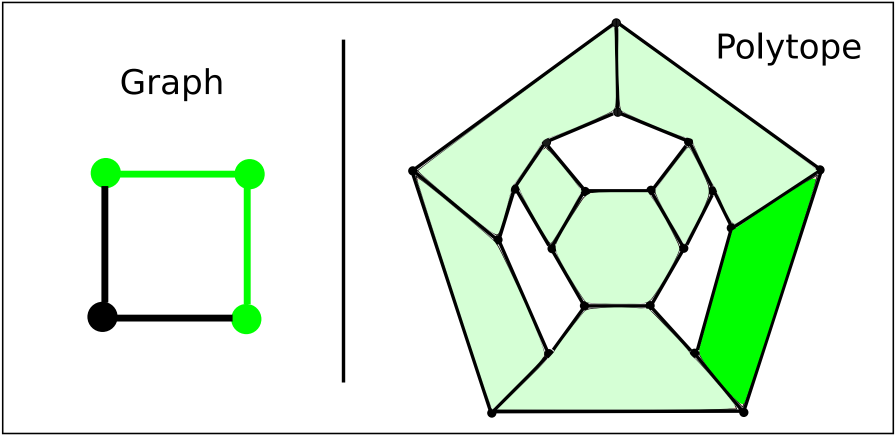
Example of a square graph.



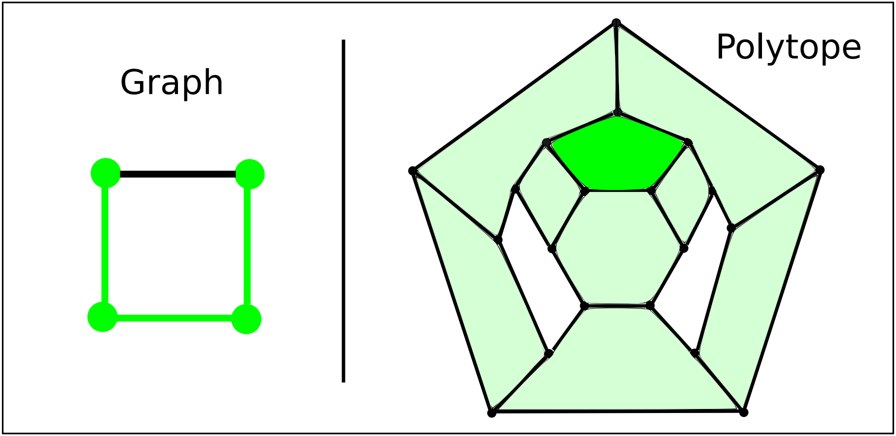
Example of a square graph.



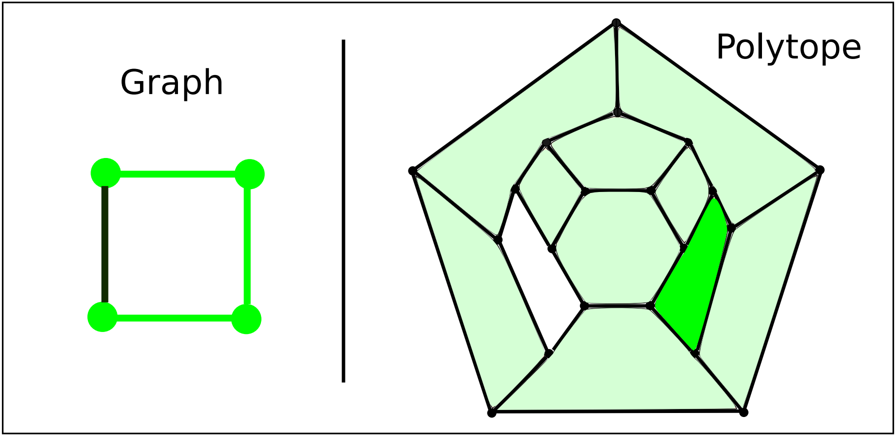
Example of a square graph.



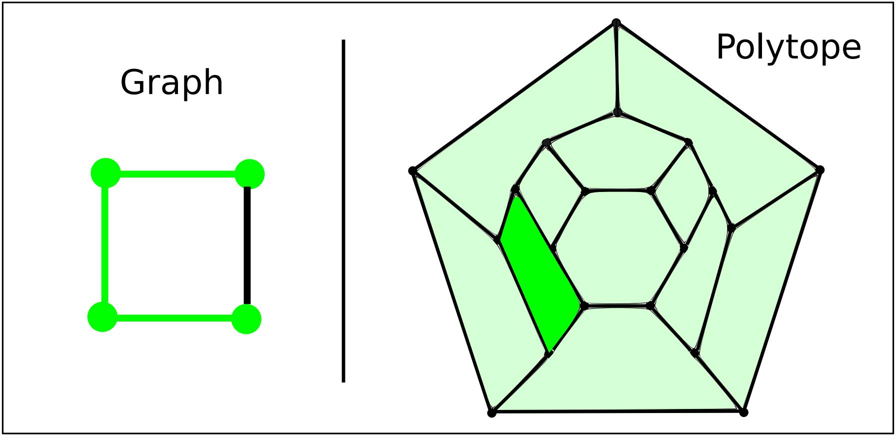
Example of a square graph.



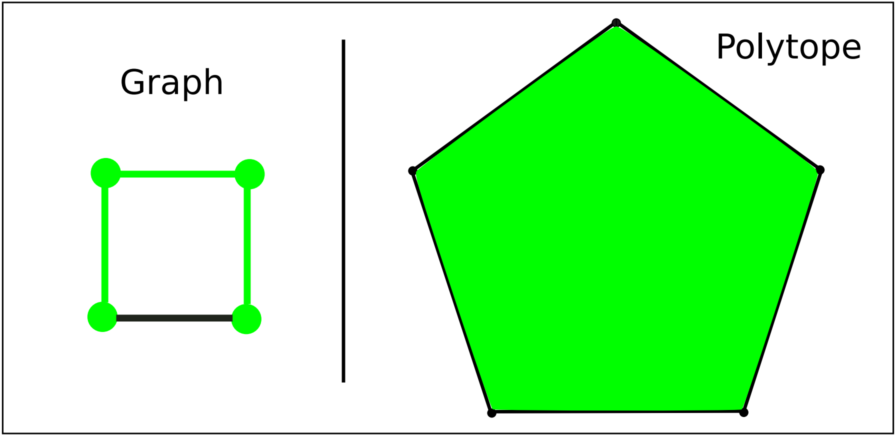
Example of a square graph.



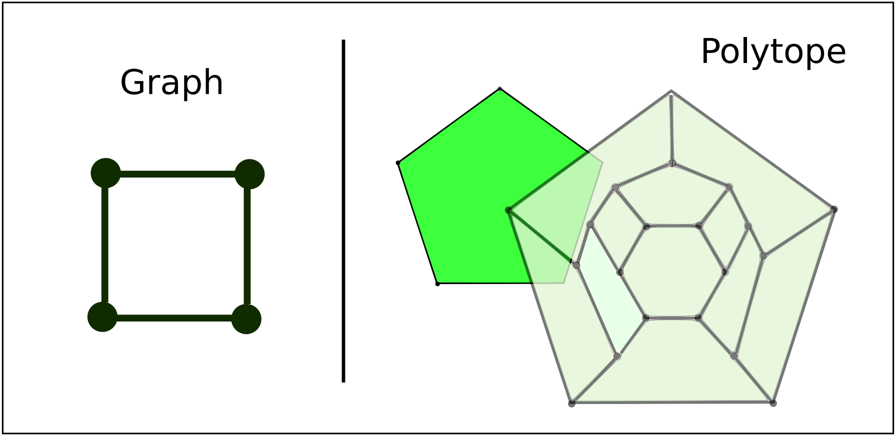
Example of a square graph.



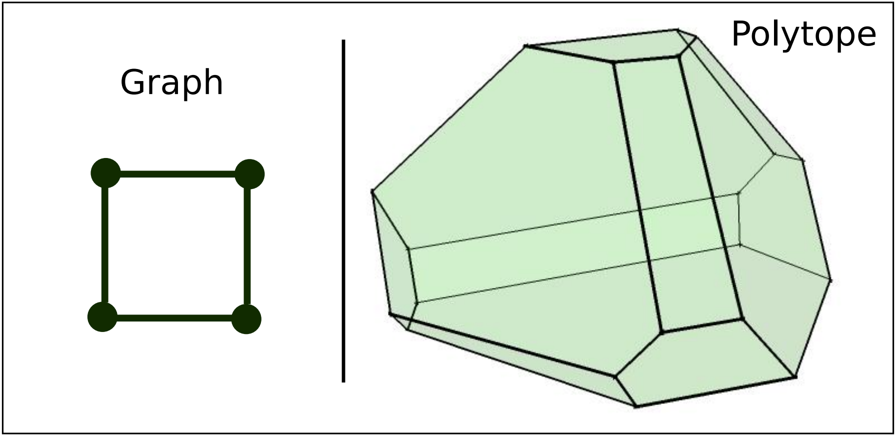
Example of a square graph.



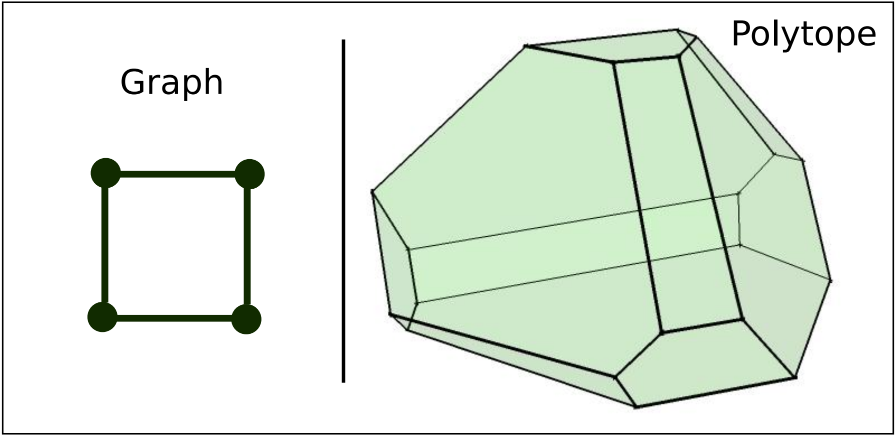
Example of a square graph.



Example of a square graph.



Example of a square graph.



Let's call these polytopes **Bracketohedra**.

An analogy

How do we use this generalization?

An analogy

How do we use this generalization?

	then
Algebraic structure	Associativity
Combinatorics	Multiply along a line
Polytopes	Associahedra
Homotopy Transfer	via A_∞-algebras
use to study	Topological spaces

An analogy

How do we use this generalization?

	then	now
Algebraic structure	Associativity	
Combinatorics	Multiply along a line	Multiply along a graph
Polytopes	Associahedra	Bracketohedra
Homotopy Transfer	via A_∞-algebras	
use to study	Topological spaces	

An analogy

How do we use this generalization?

	then	now
Algebraic structure	Associativity	
Combinatorics	Multiply along a line	Multiply along a graph
Polytopes	Associahedra	Bracketohedra
Homotopy Transfer	via A_∞-algebras	
use to study	Topological spaces	

Present Goal: Fill in this table.

An analogy

How do we use this generalization?

	then	now
Algebraic structure	Associativity	Modular Operad
Combinatorics	Multiply along a line	Multiply along a graph
Polytopes	Associahedra	Bracketohedra
Homotopy Transfer	via A_∞-algebras	
use to study	Topological spaces	

Present Goal: Fill in this table.

Modular operads.

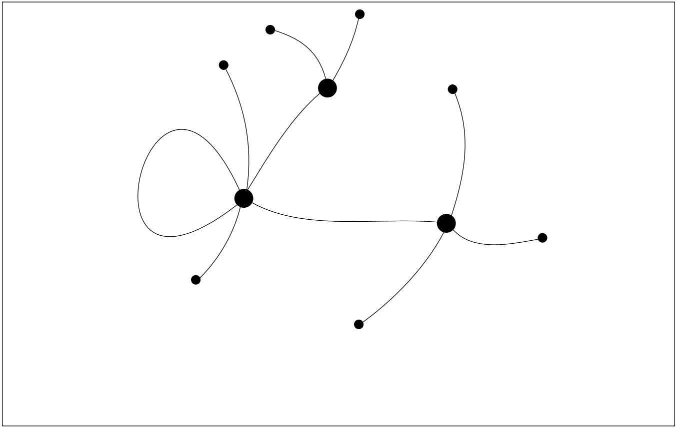
Informal **Definition:** A modular operad is a sequence of objects $(M_2, M_3, M_4, \dots)$

Modular operads.

Informal **Definition:** A modular operad is a sequence of objects $(M_2, M_3, M_4, \ldots)$ and an algebraic operation for every graph:

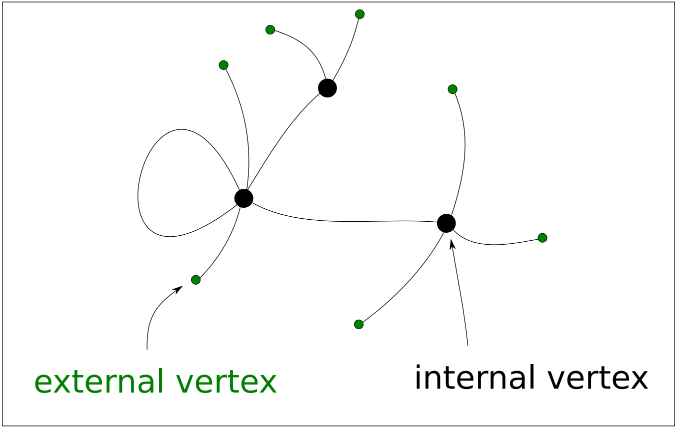
Modular operads.

Informal **Definition:** A modular operad is a sequence of objects $(M_2, M_3, M_4, \dots)$ and an algebraic operation for every graph:



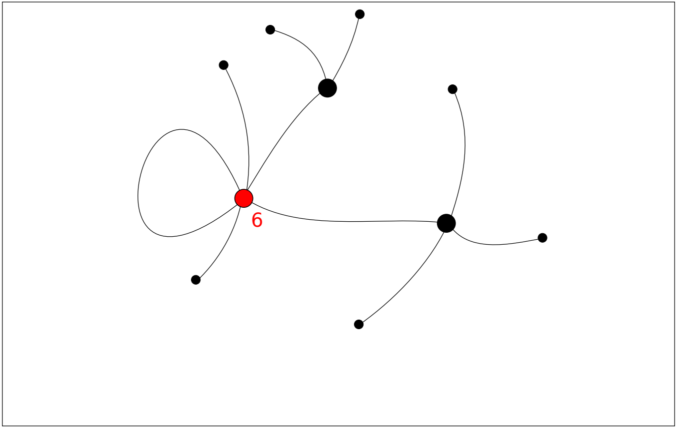
Modular operads.

Informal **Definition:** A modular operad is a sequence of objects $(M_2, M_3, M_4, \dots)$ and an algebraic operation for every graph:



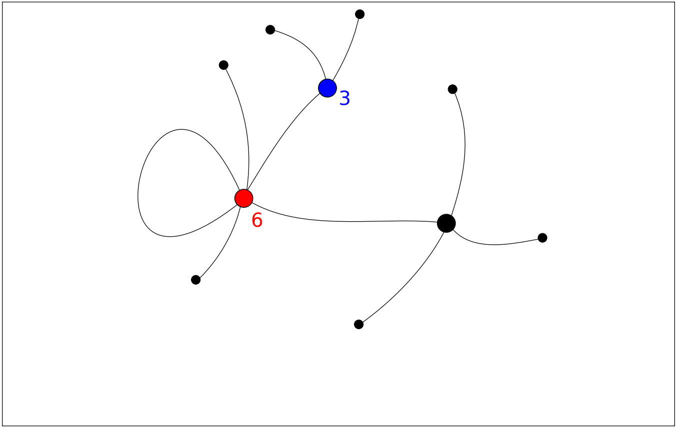
Modular operads.

Informal **Definition:** A modular operad is a sequence of objects $(M_2, M_3, M_4, \dots)$ and an algebraic operation for every graph:



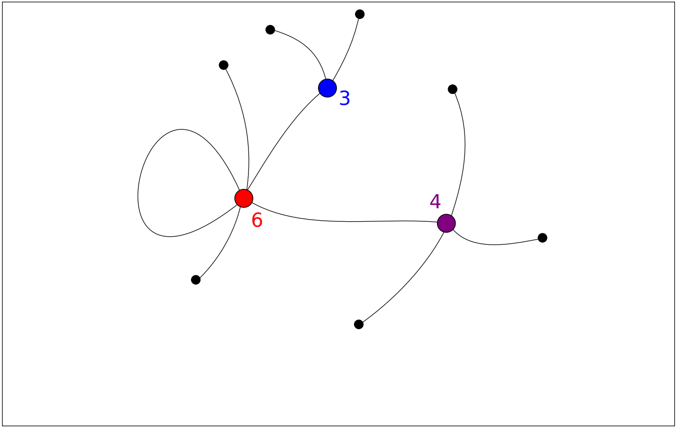
Modular operads.

Informal **Definition:** A modular operad is a sequence of objects $(M_2, M_3, M_4, \dots)$ and an algebraic operation for every graph:



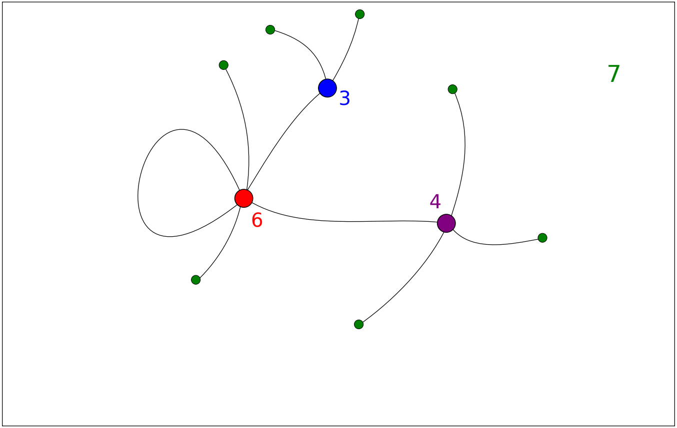
Modular operads.

Informal **Definition:** A modular operad is a sequence of objects $(M_2, M_3, M_4, \dots)$ and an algebraic operation for every graph:



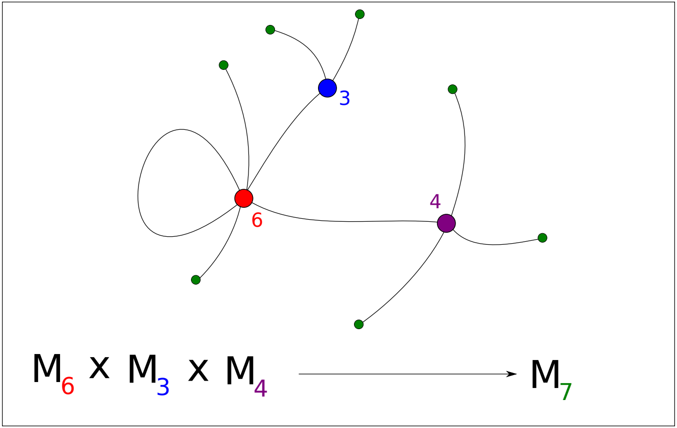
Modular operads.

Informal **Definition:** A modular operad is a sequence of objects $(M_2, M_3, M_4, \dots)$ and an algebraic operation for every graph:



Modular operads.

Informal **Definition:** A modular operad is a sequence of objects $(M_2, M_3, M_4, \dots)$ and an algebraic operation for every graph:

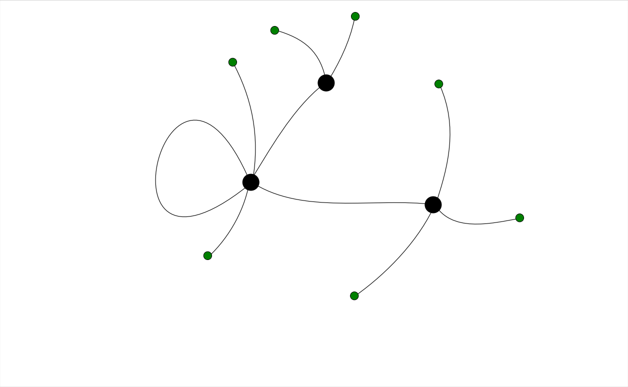


Examples of modular operads

Let A be an associative algebra and define $(M_2, M_3, M_4, ...) = (A, 0, 0, ...)$

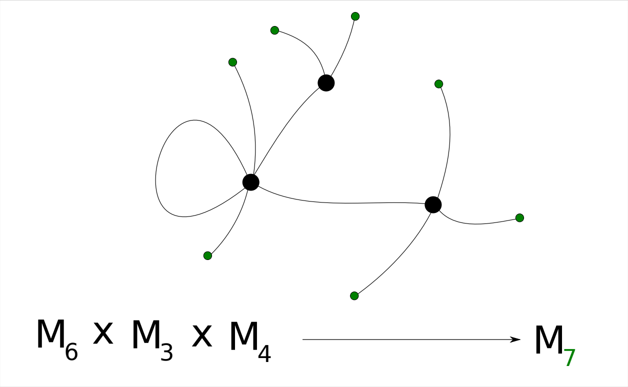
Examples of modular operads

Let A be an associative algebra and define $(M_2, M_3, M_4, ...) = (A, 0, 0, ...)$



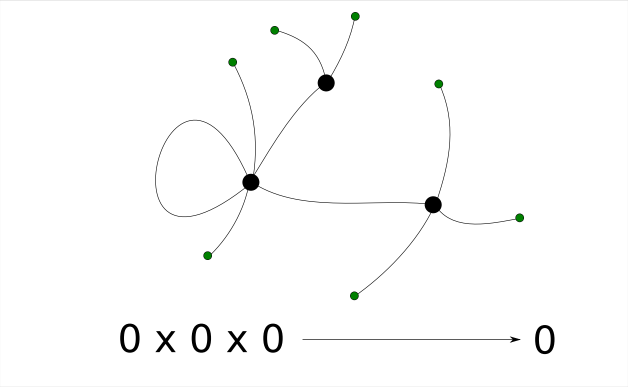
Examples of modular operads

Let A be an associative algebra and define $(M_2, M_3, M_4, ...) = (A, 0, 0, ...)$



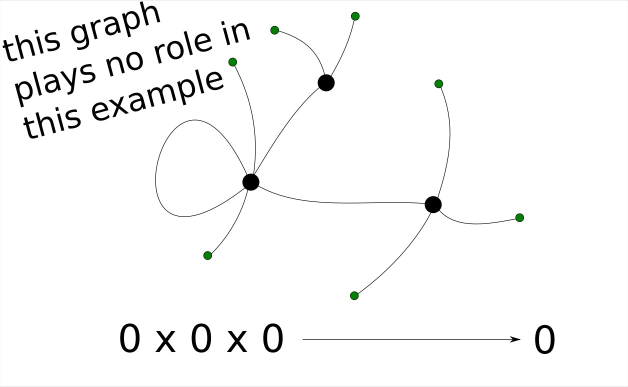
Examples of modular operads

Let A be an associative algebra and define $(M_2, M_3, M_4, \ldots) = (A, 0, 0, \ldots)$



Examples of modular operads

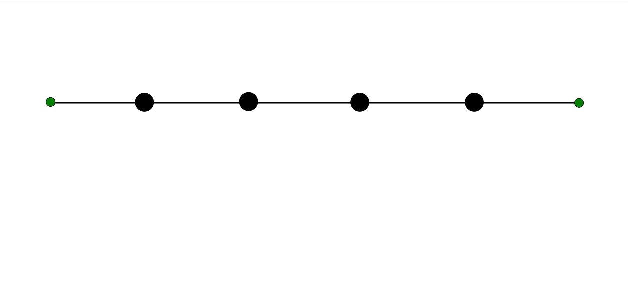
Let A be an associative algebra and define $(M_2, M_3, M_4, ...) = (A, 0, 0, ...)$



Examples of modular operads

Let A be an associative algebra and define $(M_2, M_3, M_4, ...) = (A, 0, 0, ...)$

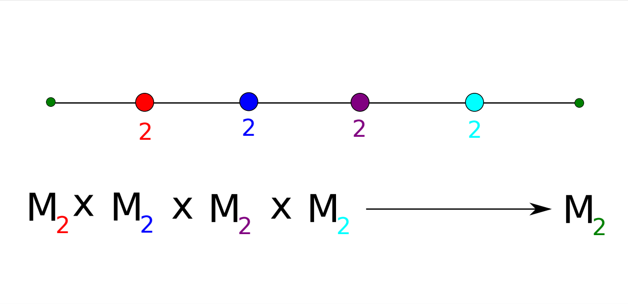
- The operations are trivial unless all internal vertices have valence 2.



Examples of modular operads

Let A be an associative algebra and define $(M_2, M_3, M_4, \ldots) = (A, 0, 0, \ldots)$

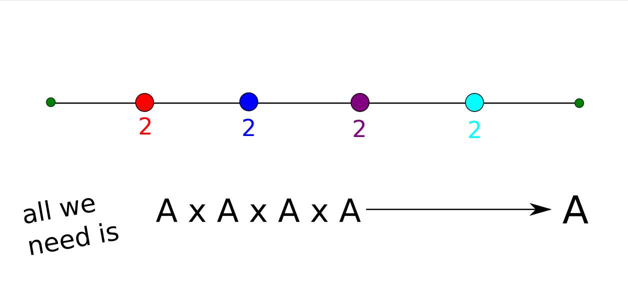
- The operations are trivial unless all internal vertices have valence 2.



Examples of modular operads

Let A be an associative algebra and define $(M_2, M_3, M_4, ...) = (A, 0, 0, ...)$

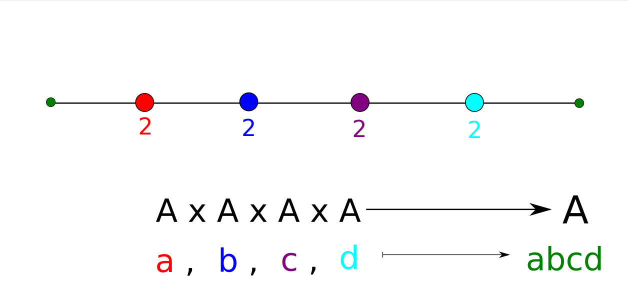
- The operations are trivial unless all internal vertices have valence 2.



Examples of modular operads

Let A be an associative algebra and define $(M_2, M_3, M_4, \ldots) = (A, 0, 0, \ldots)$

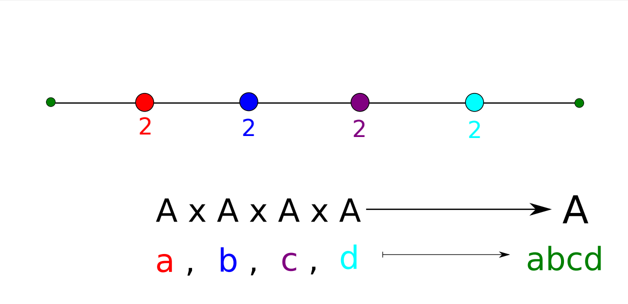
- The operations are trivial unless all internal vertices have valence 2.



Examples of modular operads

Let A be an associative algebra and define $(M_2, M_3, M_4, \ldots) = (A, 0, 0, \ldots)$

- The operations are trivial unless all internal vertices have valence 2.



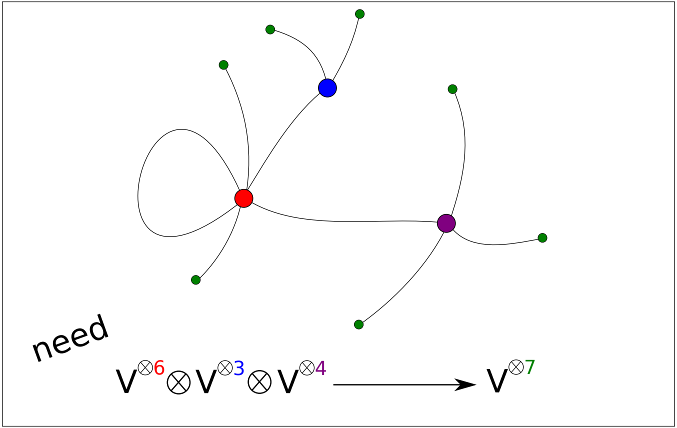
- Modular operads generalize associativity.

Examples of modular operads

Let V be a vector space and $V \otimes V \xrightarrow{\langle -,- \rangle} \mathbb{Q}$ an inner product.
Define $(M_2, M_3, M_4, ...) = (V^{\otimes 2}, V^{\otimes 3}, V^{\otimes 4}, ...)$.

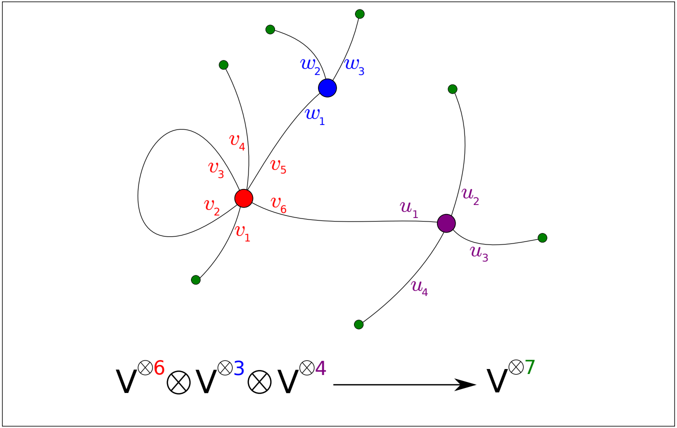
Examples of modular operads

Let V be a vector space and $V \otimes V \xrightarrow{\langle -,- \rangle} \mathbb{Q}$ an inner product.
Define $(M_2, M_3, M_4, \ldots) = (V^{\otimes 2}, V^{\otimes 3}, V^{\otimes 4}, \ldots)$.



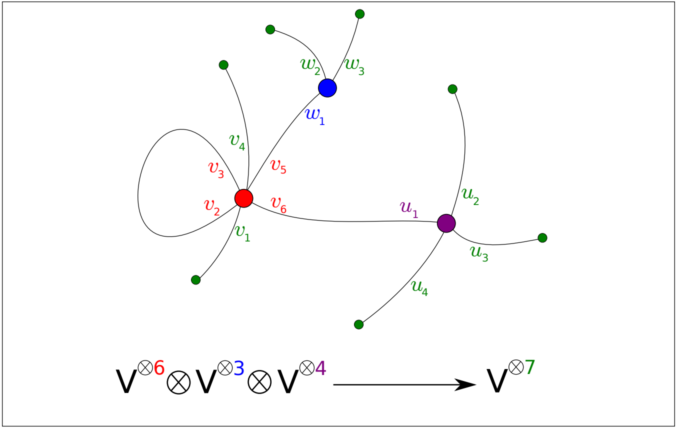
Examples of modular operads

Let V be a vector space and $V \otimes V \xrightarrow{\langle -,- \rangle} \mathbb{Q}$ an inner product.
Define $(M_2, M_3, M_4, \dots) = (V^{\otimes 2}, V^{\otimes 3}, V^{\otimes 4}, \dots)$.



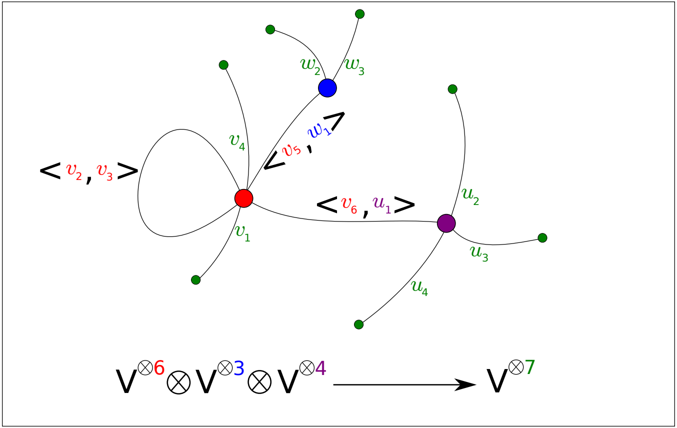
Examples of modular operads

Let V be a vector space and $V \otimes V \xrightarrow{\langle -,- \rangle} \mathbb{Q}$ an inner product.
Define $(M_2, M_3, M_4, \ldots) = (V^{\otimes 2}, V^{\otimes 3}, V^{\otimes 4}, \ldots)$.



Examples of modular operads

Let V be a vector space and $V \otimes V \xrightarrow{\langle -,- \rangle} \mathbb{Q}$ an inner product.
Define $(M_2, M_3, M_4, \ldots) = (V^{\otimes 2}, V^{\otimes 3}, V^{\otimes 4}, \ldots)$.

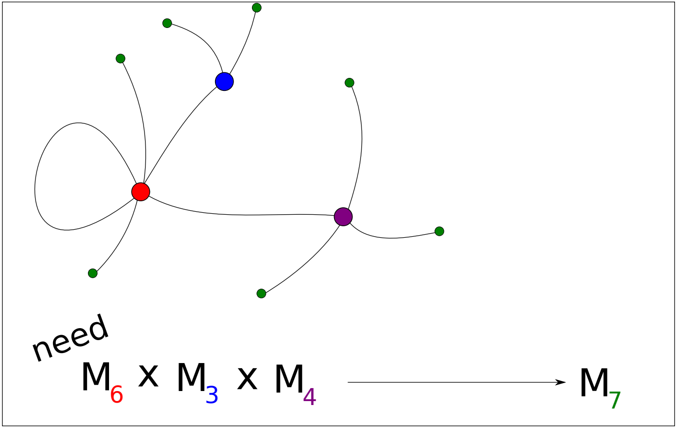


Examples of modular operads

Surfaces: Let M_n be the set of compact, orientable surfaces with n boundary components.

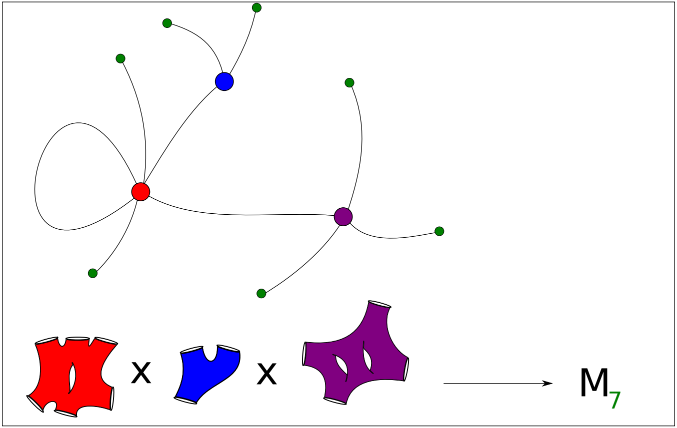
Examples of modular operads

Surfaces: Let M_n be the set of compact, orientable surfaces with n boundary components.



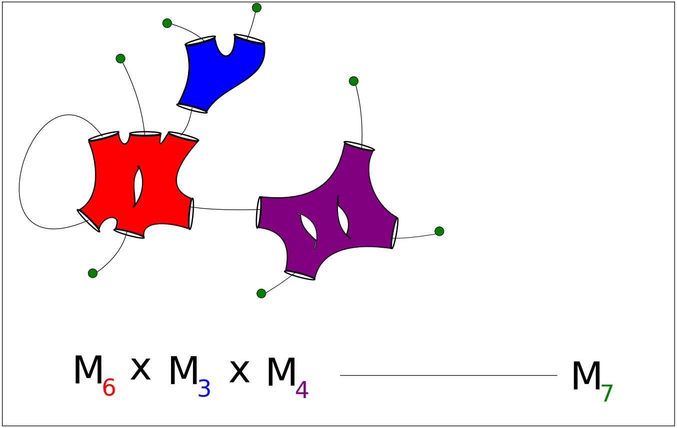
Examples of modular operads

Surfaces: Let M_n be the set of compact, orientable surfaces with n boundary components.



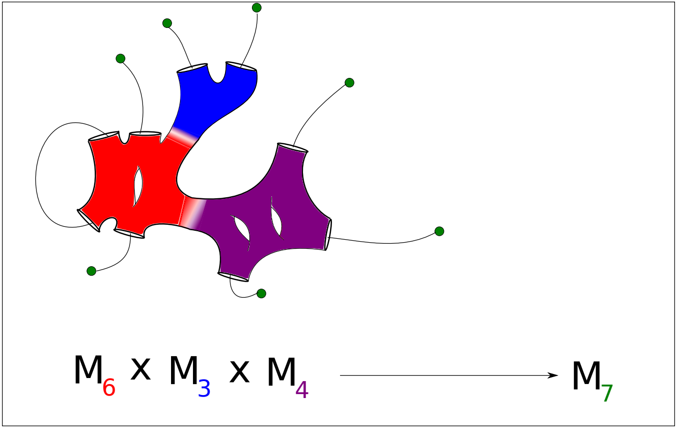
Examples of modular operads

Surfaces: Let M_n be the set of compact, orientable surfaces with n boundary components.



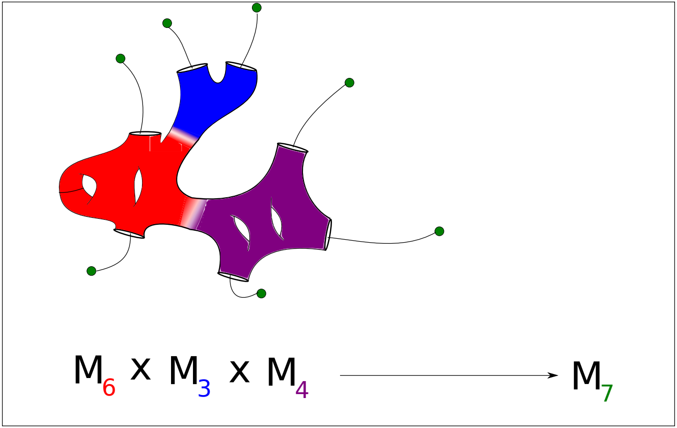
Examples of modular operads

Surfaces: Let M_n be the set of compact, orientable surfaces with n boundary components.



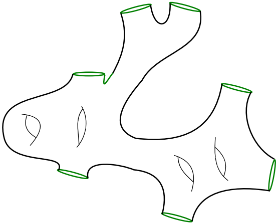
Examples of modular operads

Surfaces: Let M_n be the set of compact, orientable surfaces with n boundary components.



Examples of modular operads

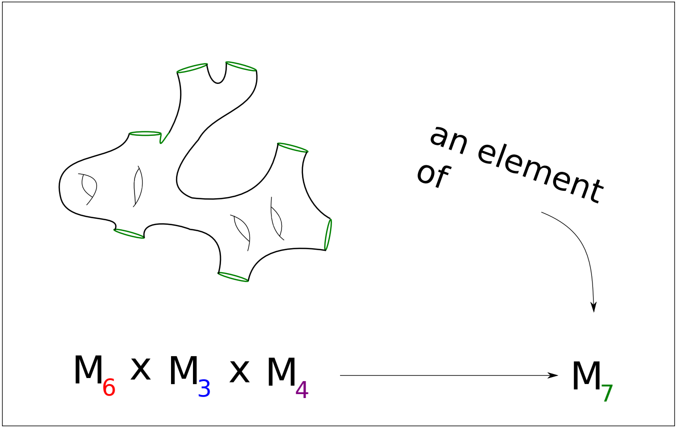
Surfaces: Let M_n be the set of compact, orientable surfaces with n boundary components.



$$M_{6} \times M_{3} \times M_{4} \longrightarrow M_{7}$$

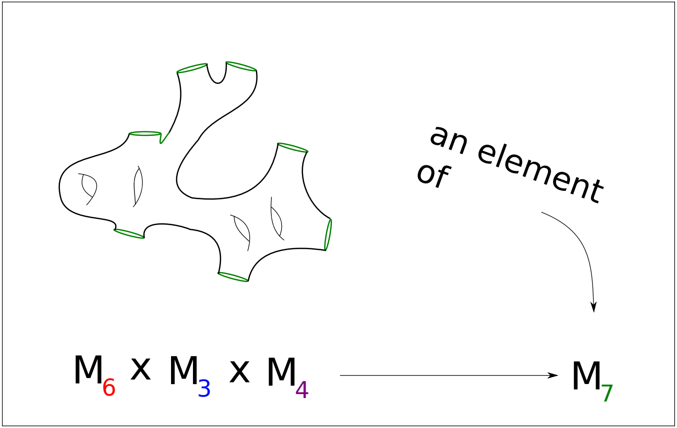
Examples of modular operads

Surfaces: Let M_n be the set of compact, orientable surfaces with n boundary components.



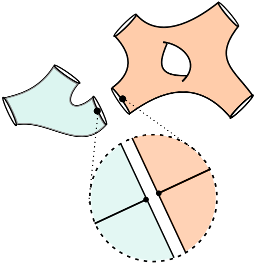
Examples of modular operads

Surfaces: Let M_n be the set of compact, orientable surfaces with n boundary components.

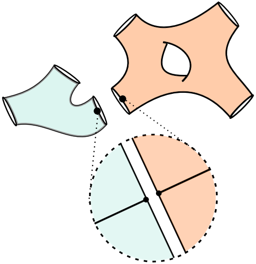


- Surfaces form a modular operad by gluing.

Other examples of modular operads:

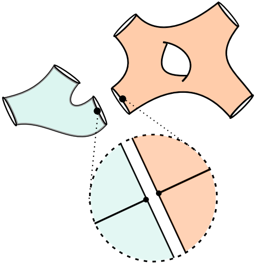


Other examples of modular operads:



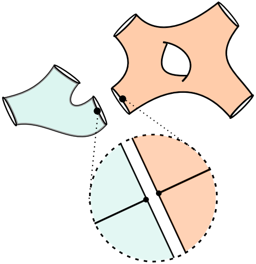
- Moduli spaces of surfaces with boundary.

Other examples of modular operads:



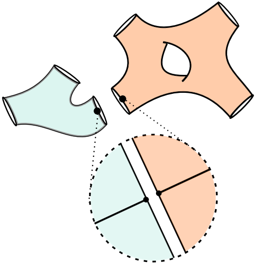
- Moduli spaces of surfaces with boundary.
- Deligne-Mumford compactifications of surfaces with punctures.

Other examples of modular operads:



- Moduli spaces of surfaces with boundary.
- Deligne-Mumford compactifications of surfaces with punctures.
- Graph complexes...

Other examples of modular operads:



- Moduli spaces of surfaces with boundary.
- Deligne-Mumford compactifications of surfaces with punctures.
- Graph complexes...

It's preferable to separate out the genus: $\mathcal{M} = \{\mathcal{M}_{g,n}\}$.

Back to the analogy

	then	now
Algebraic structure	Associativity	Modular Operad
Combinatorics	Multiply along a line	Multiply along a graph
Polytopes	Associahedra	Bracketohedra
Homotopy Transfer	via A_∞-algebras	
use to study	Topological spaces	

Present Goal: Fill in this table.

Back to the analogy

	then	now
Algebraic structure	Associativity	Modular Operad
Combinatorics	Multiply along a line	Multiply along a graph
Polytopes	Associahedra	Bracketohedra
Homotopy Transfer	via A_∞-algebras	theorem
use to study	Topological spaces	

Present Goal: Fill in this table.

Homotopy Transfer Theorem for Modular Operads.

Generalizing the classical A_∞ story we have:

Homotopy Transfer Theorem for Modular Operads.

Generalizing the classical A_∞ story we have:

Theorem (W.)

Every dg modular operad is quasi-isomorphic to a "A_∞-modular operad" structure on its homology.

Homotopy Transfer Theorem for Modular Operads.

Generalizing the classical A_∞ story we have:

Theorem (W.)

Every dg modular operad is quasi-isomorphic to a "A_∞-modular operad" structure on its homology.

Proof.

1. Encode modular operads as algebras over a *quadratic operad*.

Homotopy Transfer Theorem for Modular Operads.

Generalizing the classical A_∞ story we have:

Theorem (W.)

Every dg modular operad is quasi-isomorphic to a "A_∞-modular operad" structure on its homology.

Proof.

1. Encode modular operads as algebras over a *quadratic operad*. This requires using colored operads whose colors form not just a set but a groupoid.

Homotopy Transfer Theorem for Modular Operads.

Generalizing the classical A_∞ story we have:

Theorem (W.)

Every dg modular operad is quasi-isomorphic to a "A_∞-modular operad" structure on its homology.

Proof.

1. Encode modular operads as algebras over a *quadratic operad*. This requires using colored operads whose colors form not just a set but a groupoid.
2. Prove that the operad encoding modular operads is Koszul.

Homotopy Transfer Theorem for Modular Operads.

Generalizing the classical A_∞ story we have:

Theorem (W.)

Every dg modular operad is quasi-isomorphic to a "A_∞-modular operad" structure on its homology.

Proof.

1. Encode modular operads as algebras over a *quadratic operad*. This requires using colored operads whose colors form not just a set but a groupoid.
2. Prove that the operad encoding modular operads is Koszul. This is where we use contractibility of bracketohedra.

Homotopy Transfer Theorem for Modular Operads.

Generalizing the classical A_∞ story we have:

Theorem (W.)

Every dg modular operad is quasi-isomorphic to a "A_∞-modular operad" structure on its homology.

Proof.

1. Encode modular operads as algebras over a *quadratic operad*. This requires using colored operads whose colors form not just a set but a groupoid.
2. Prove that the operad encoding modular operads is Koszul. This is where we use contractibility of bracketohedra.
3. Generalize classical Koszul duality theory from operads to groupoid colored operads.

$\square$

Back to the analogy... one last time.

How do we use this generalization?

	then	now
Algebraic structure	Associativity	Modular Operad
Combinatorics	Multiply along a line	Multiply along a graph
Polytopes	Associahedra	Bracketohedra
Homotopy Transfer	via A_∞-algebras	via A_∞- modular operads
use to study	Topological spaces	

Back to the analogy... one last time.

How do we use this generalization?

	then	now
Algebraic structure	Associativity	Modular Operad
Combinatorics	Multiply along a line	Multiply along a graph
Polytopes	Associahedra	Bracketohedra
Homotopy Transfer	via A_∞-algebras	via A_∞- modular operads
use to study	Topological spaces	graph complexes

Back to the analogy... one last time.

How do we use this generalization?

	then	now
Algebraic structure	Associativity	Modular Operad
Combinatorics	Multiply along a line	Multiply along a graph
Polytopes	Associahedra	Bracketohedra
Homotopy Transfer	via A_∞-algebras	via A_∞- modular operads
use to study	Topological spaces	graph complexes
how?	Massey Products	

Back to the analogy... one last time.

How do we use this generalization?

	then	now
Algebraic structure	Associativity	Modular Operad
Combinatorics	Multiply along a line	Multiply along a graph
Polytopes	Associahedra	Bracketohedra
Homotopy Transfer	via A_∞-algebras	via A_∞- modular operads
use to study	Topological spaces	graph complexes
how?	Massey Products	Massey Products

Next time...