Introduction to Graph Complexes - III

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Last time...

An analogy:

	then	now
Algebraic structure	Associativity	Modular Operad
Combinatorics	Multiply along a line	Multiply along a graph
Polytopes	Associahedra	Bracketohedra
Homotopy Transfer	via A_∞-algebras	via A_∞- modular operads

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Let's package these higher operations using the bar construction.

Analogy continues... the bar construction

Start with a group G, let's construct a topological space.

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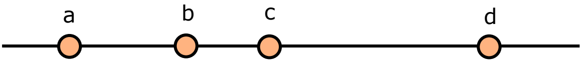

$a, b, c, d \in G$

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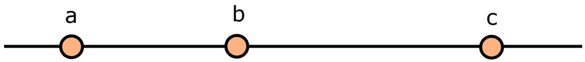

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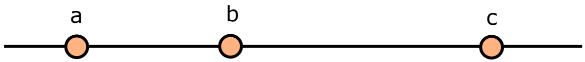

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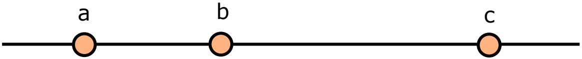

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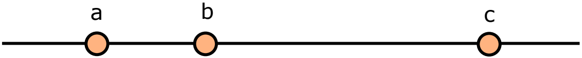

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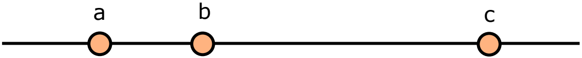

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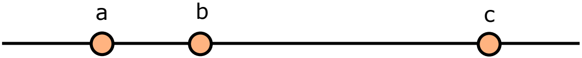

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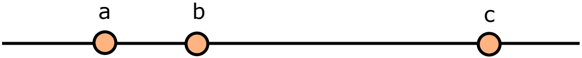

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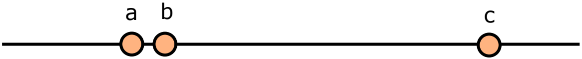

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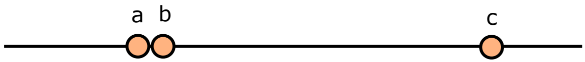

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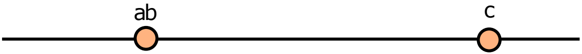

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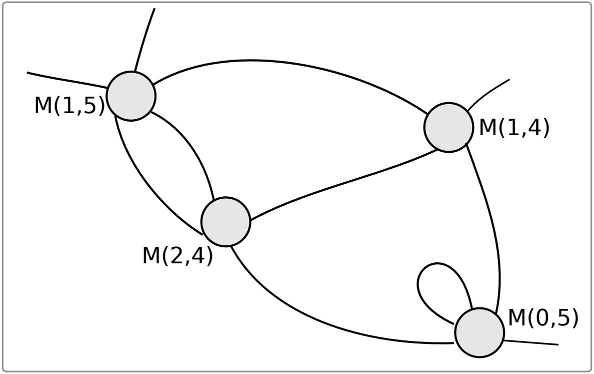
Analogously, for A an associative algebra:

$$\mathrm{B}(A) = (\oplus A^{\otimes n}, d)$$

Bar Construction for Modular Operads

Input is a modular operad M.



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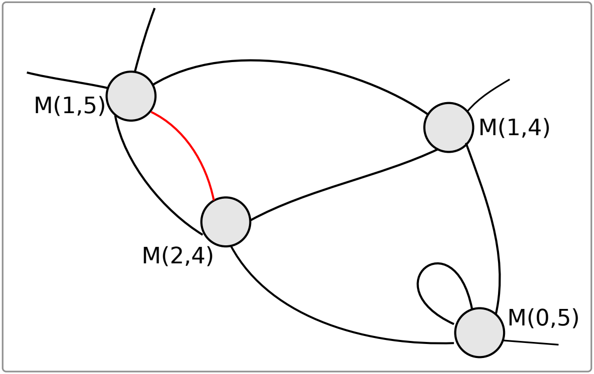

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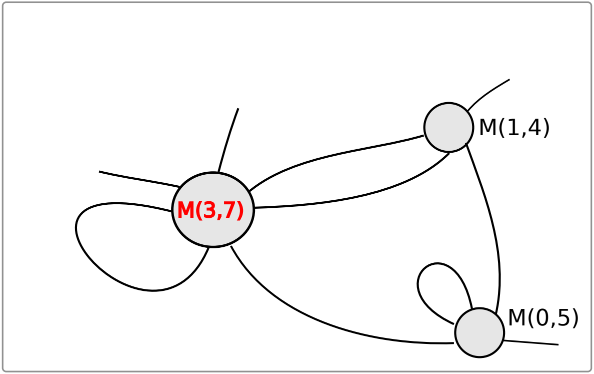

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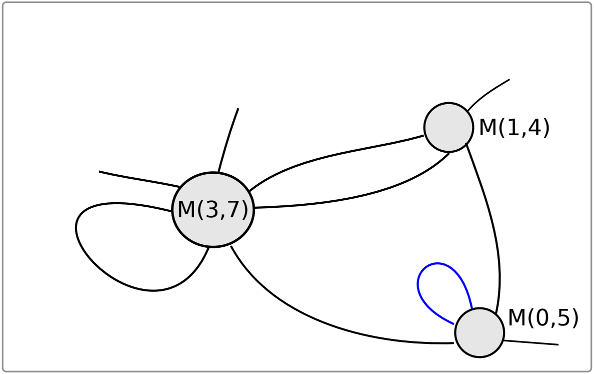

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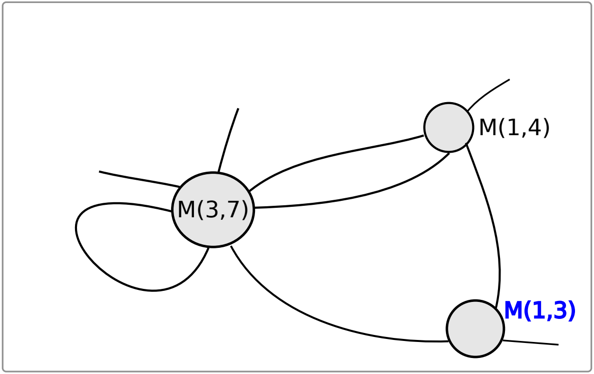

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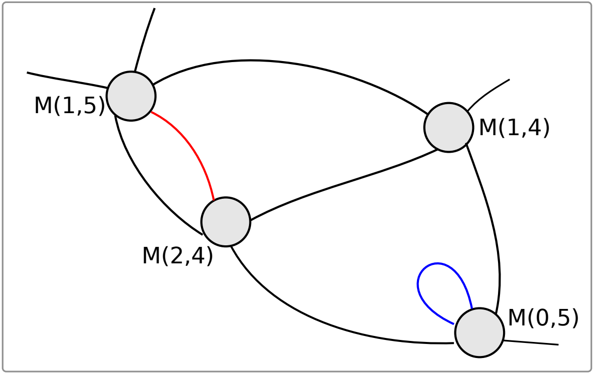

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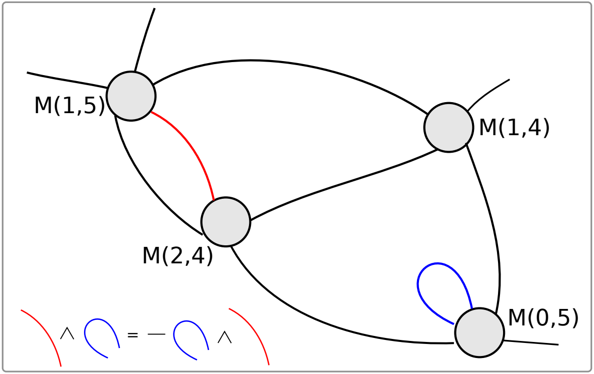

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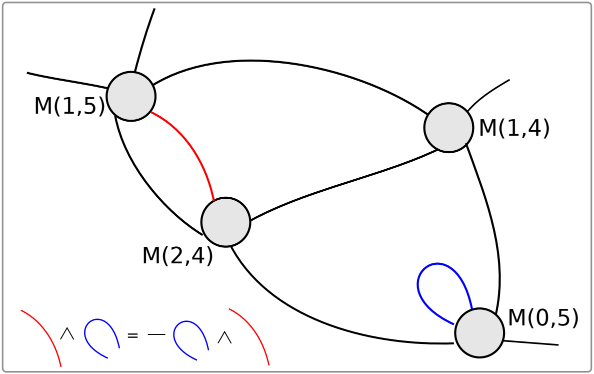

Theorem (Getzler-Kapranov)

$$FT^2(M) \sim M$$

Graph complexes via the Feynman transform

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- For any operad $\mathcal{O}$, we can consider the $\mathcal{O}$-labeled graph complex $\mathrm{FT}(\mathcal{O})$.

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- Upshot: if I didn't know the dimension of $Lie(n)$, Koszul duality would tell me how to find it (invert a power series).

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A few facts:

- Com and Lie form operads that are related by Koszul duality.

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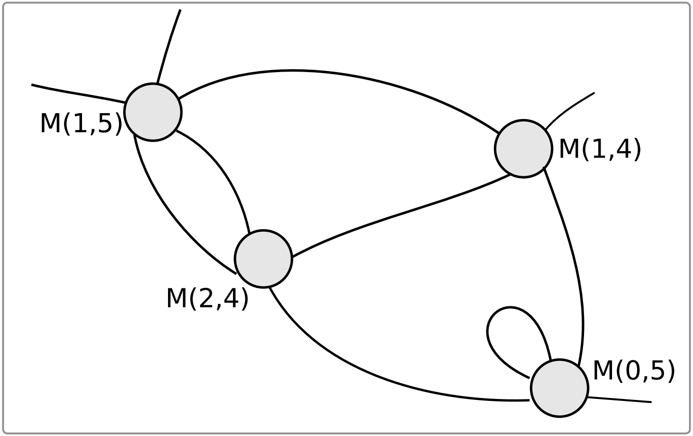
Question: How are these two variations of graph homology related?

We can give an answer using the A_∞-analog of the Feynman transform.

A Picture of the A_∞ Feynman transform

Input is an A_∞-modular operad $\mathcal{O}$.



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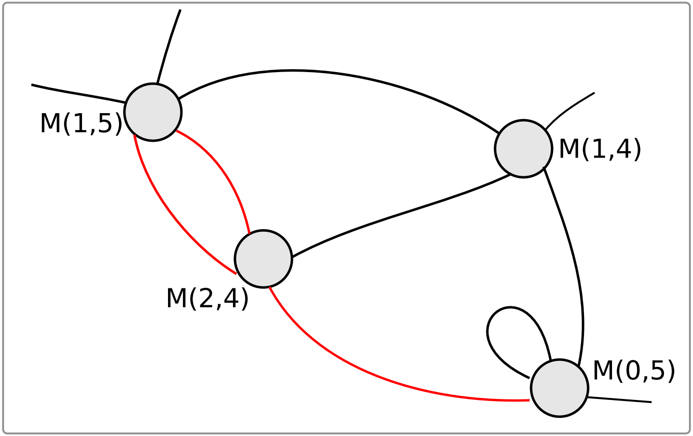

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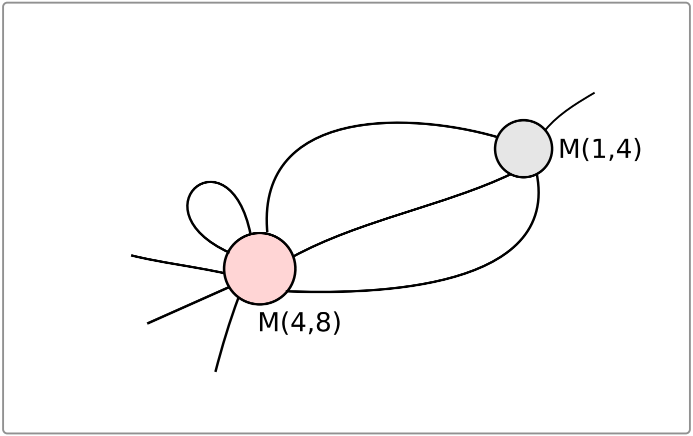

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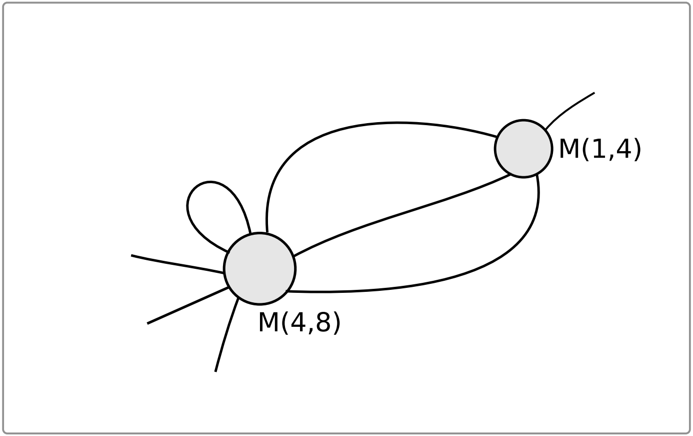

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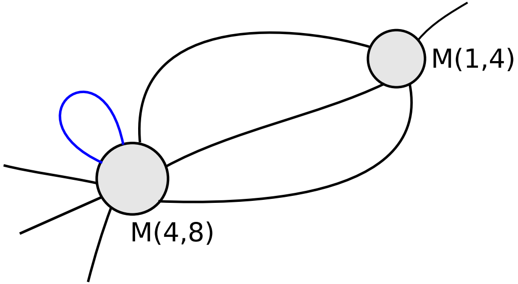

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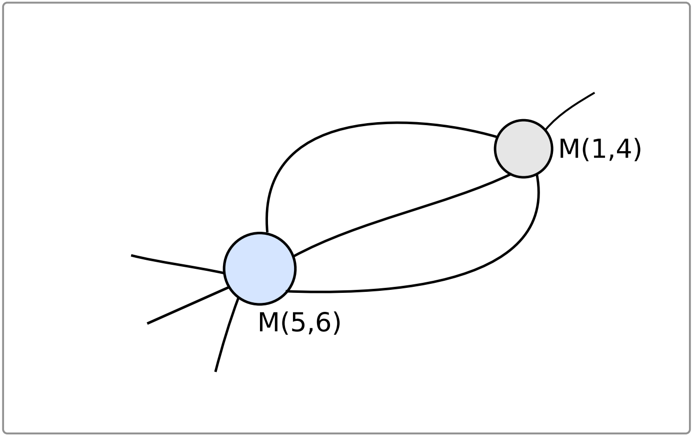

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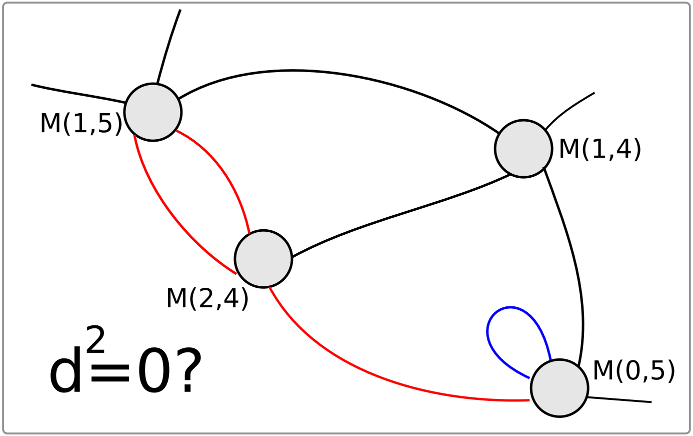

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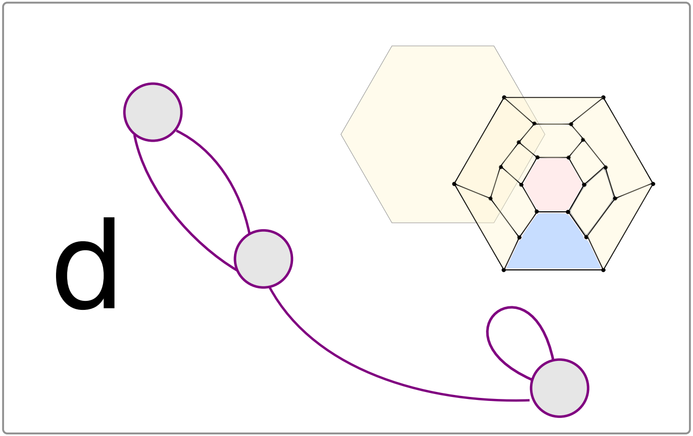

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- But the modular operad Lie is 0 in higher genus – just an operad, thus each complex $\mathrm{FT}(H_*(\Gamma))(g, n)$ with $g \geq 1$ is acyclic.

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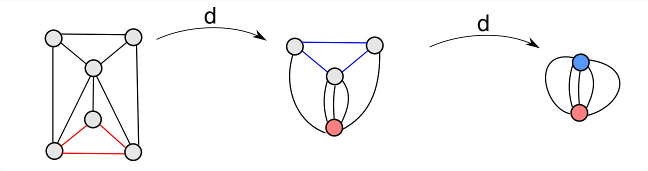

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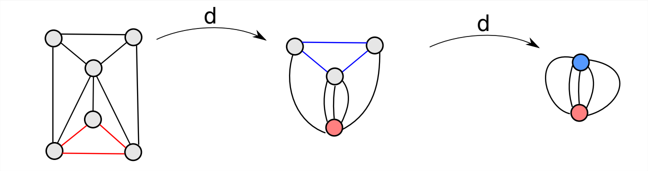

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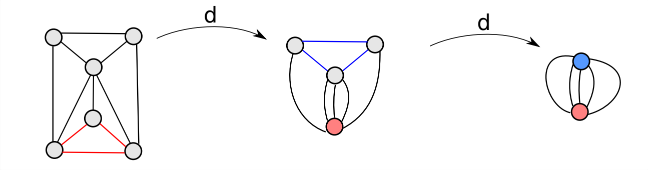
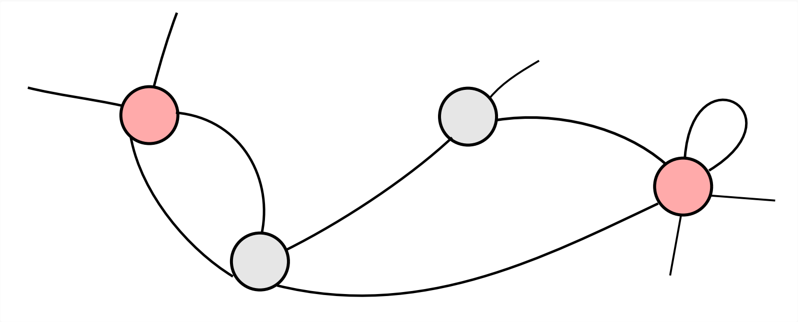


Thus every homology class $\mathrm{ft}(\mathrm{Com})(g, n)$ is a represented in $\mathrm{ft}(H(\Gamma))(g, n)$, via Massey products.

The acyclic graph complex – $\mathrm{ft}(H(\Gamma))$



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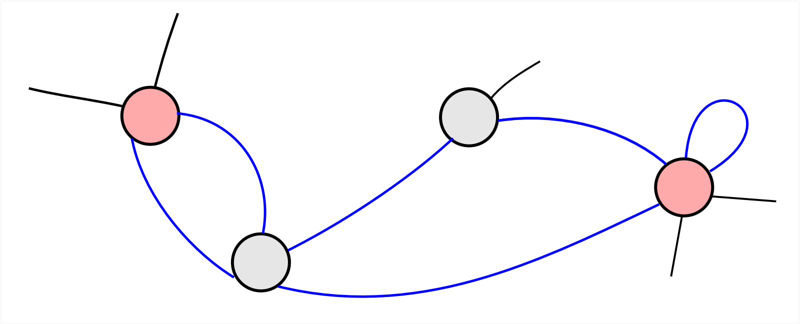

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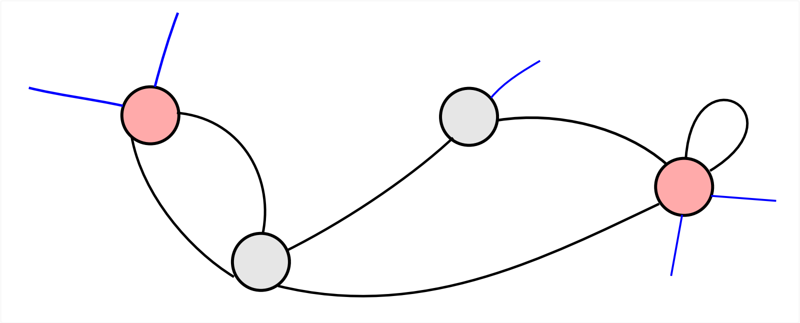

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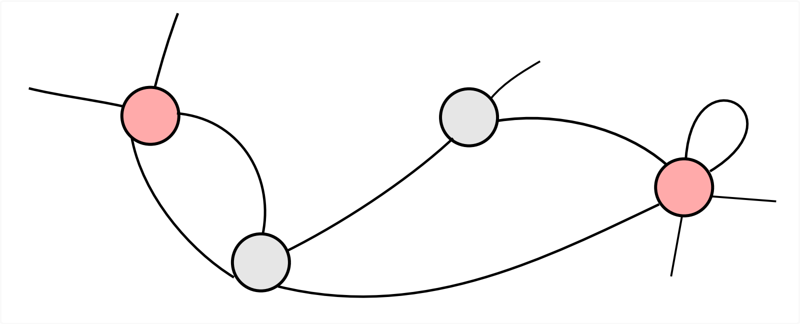

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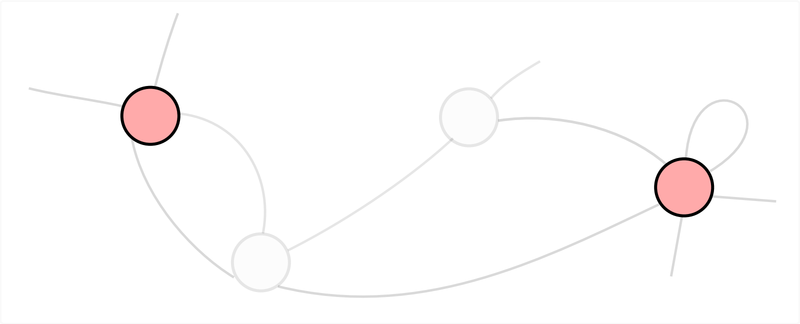

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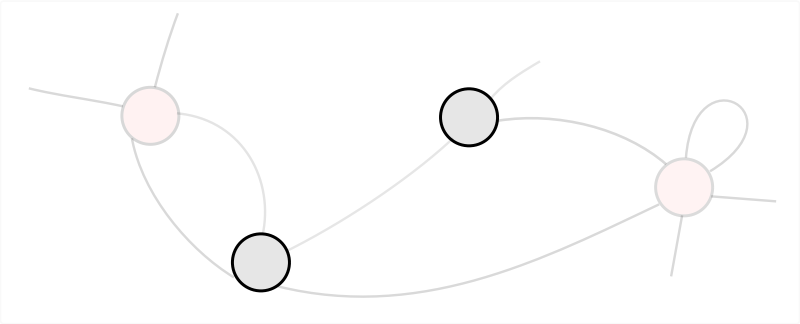

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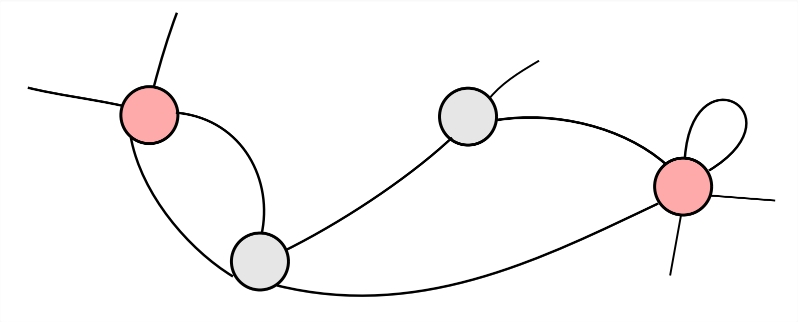

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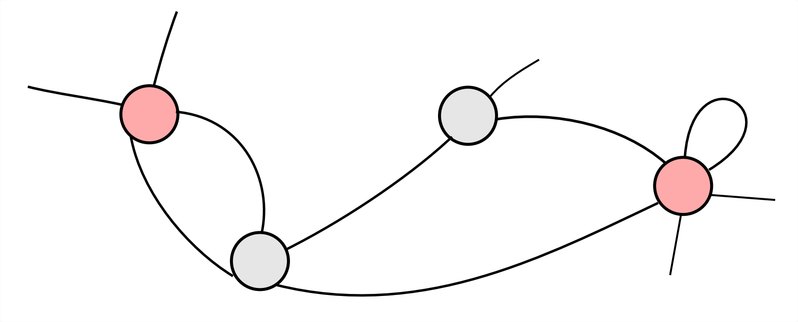

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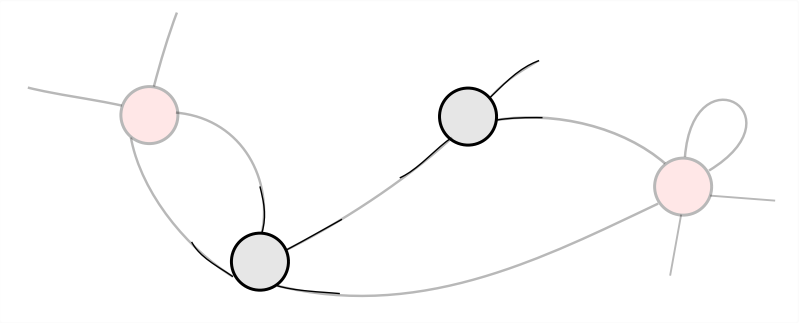

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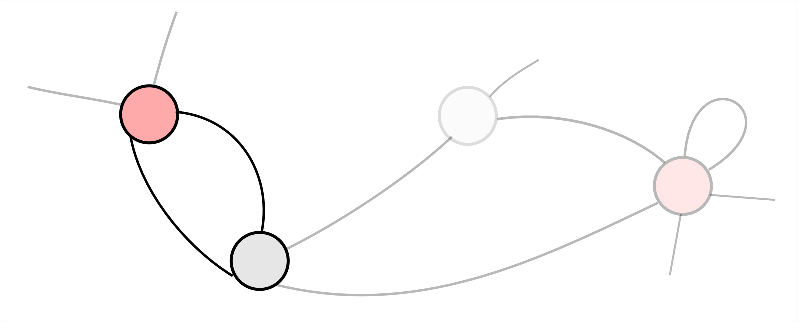

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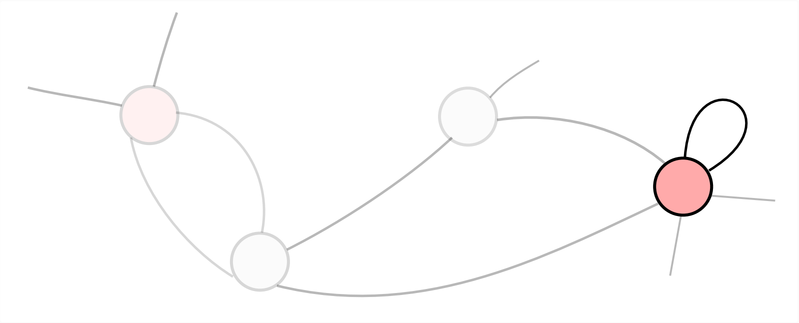

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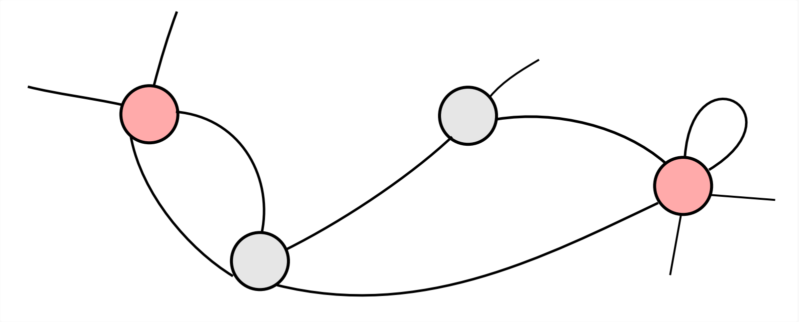

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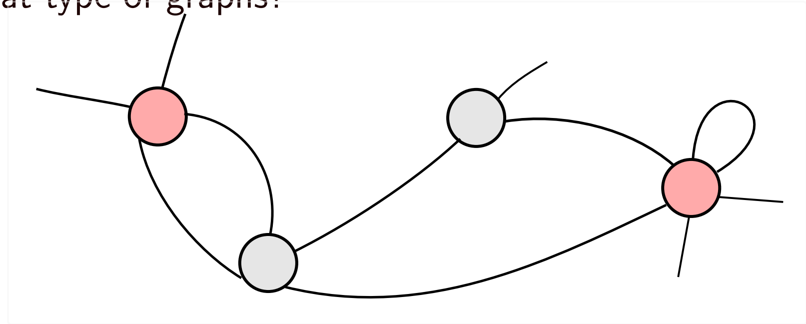

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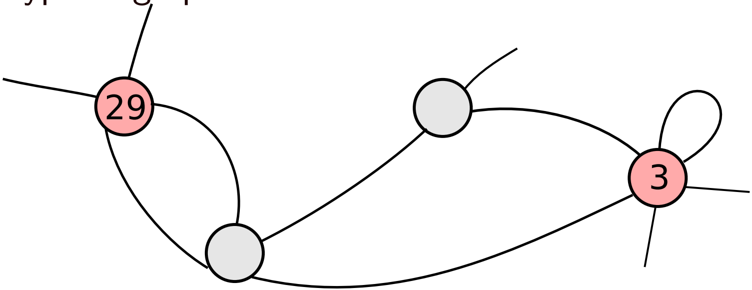

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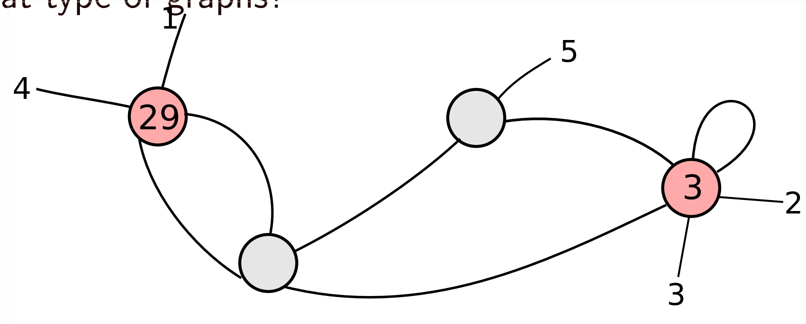

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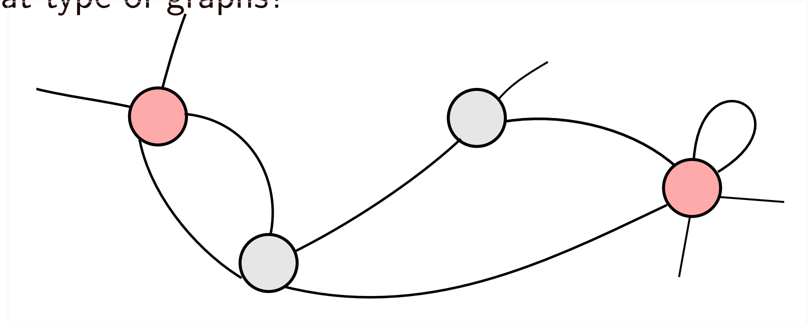

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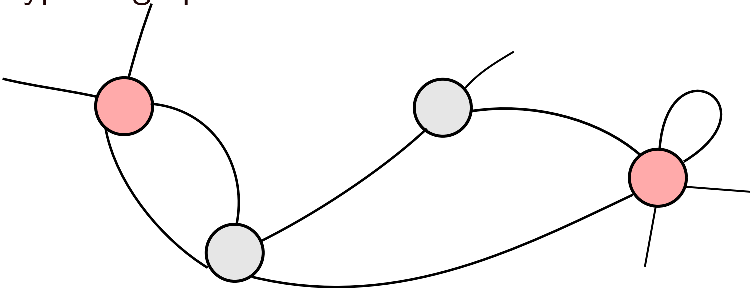

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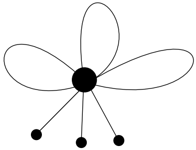
- Red vertices by higher homology classes.

Lie graph homology

Let $X_{g,n}$ be a wedge of g circles and n pointed intervals



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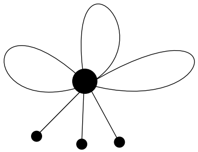

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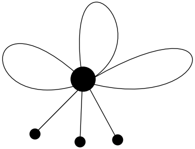

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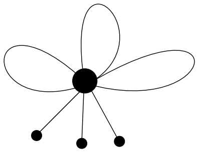

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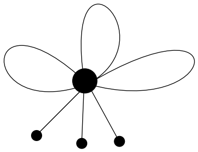

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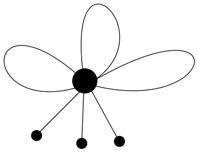

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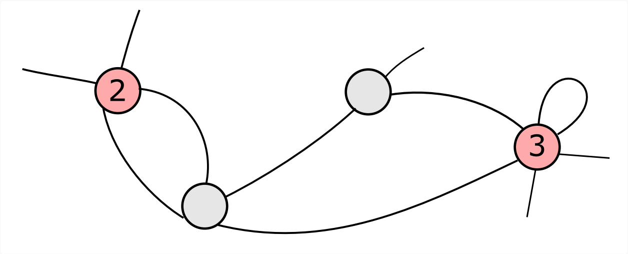
- $H_{2j}(\Gamma_{1,2j+1})$ is the alternating representation.

The acyclic graph complex – $\mathrm{ft}(H(\Gamma))$

To each graph γ, form a graded vector space

$$\mathrm{ft}(H(\Gamma))(g,n) = \bigoplus_{\gamma} \Big[\bigotimes_{\substack{\text{red vertices} \\ \text{of } \gamma}} \widetilde{H}_*(\Gamma_{g(v),n(v)}) \Big]$$



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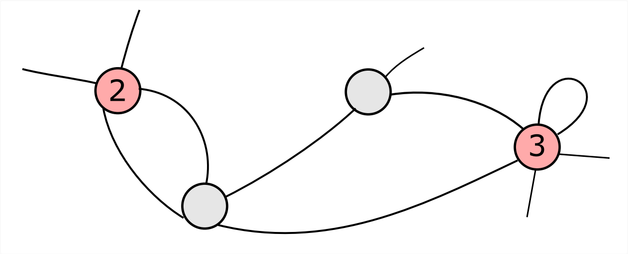

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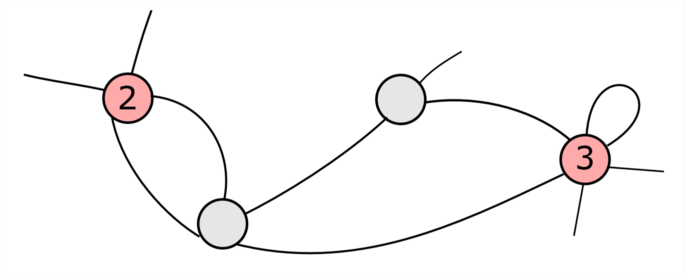

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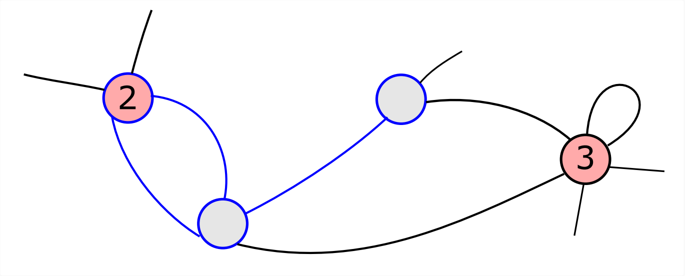

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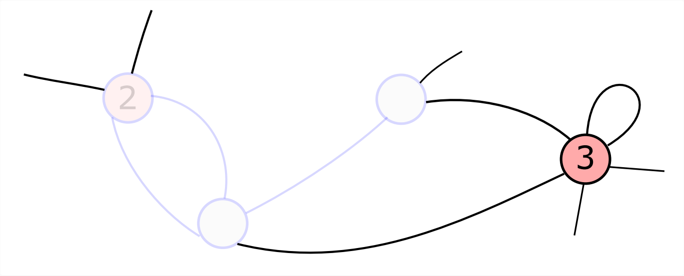

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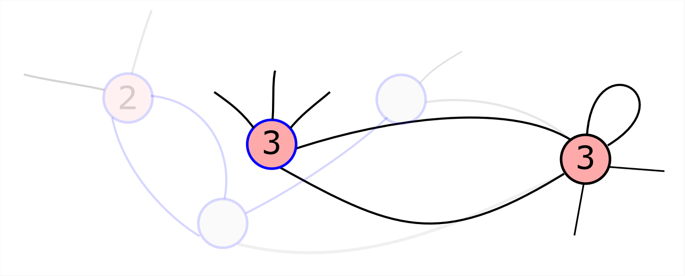

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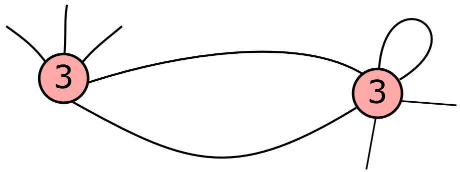

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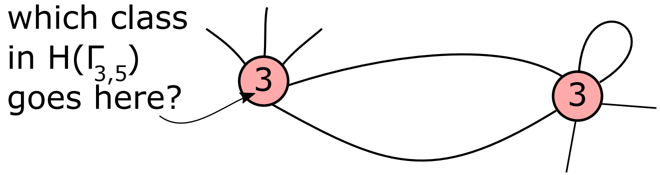

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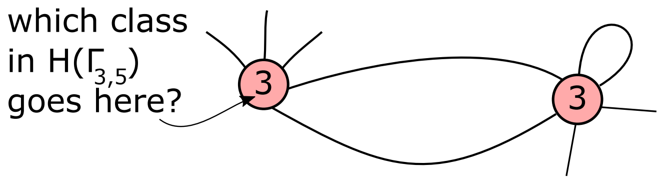

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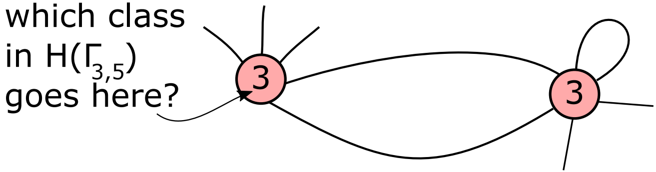

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... a Massey product.

The short exact sequence

Let R be the span of graphs having at least one red vertex.
For each (g, n) with $g \geq 1$ there is a short exact sequence

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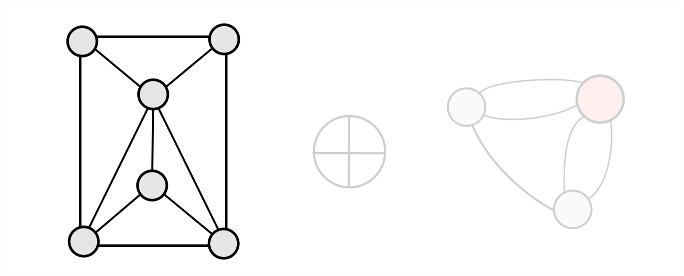

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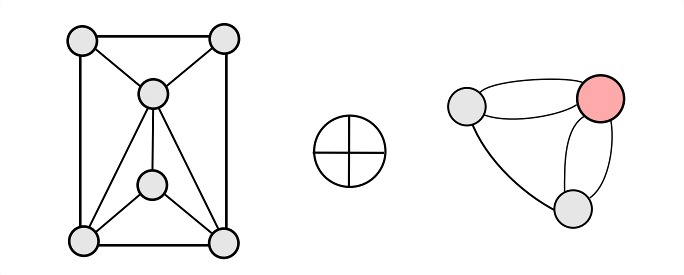

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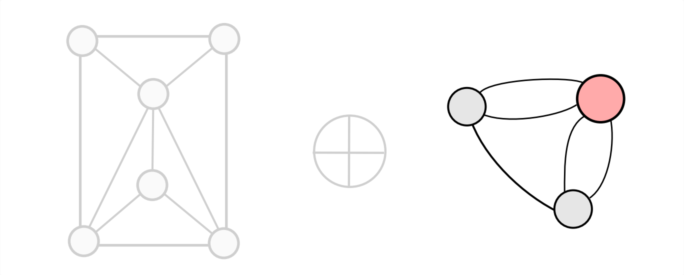

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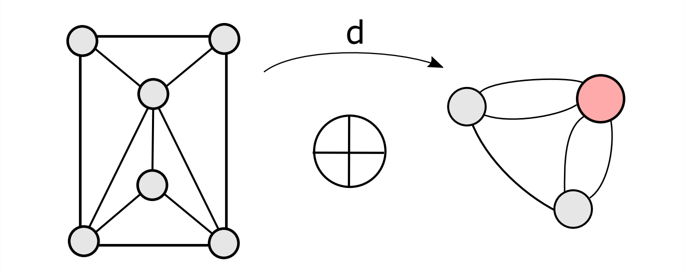

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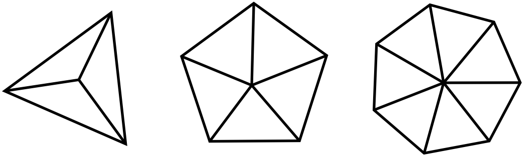

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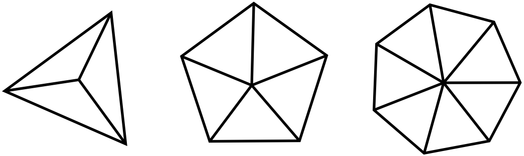

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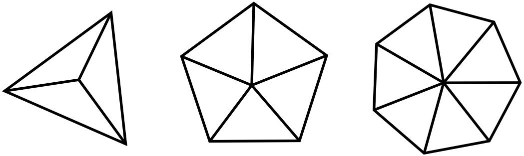

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The class α_{2j+1} *is in the image of the Massey product associated to the stable* $2j+1$*-gon.*

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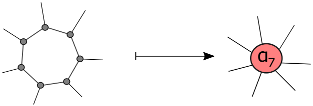

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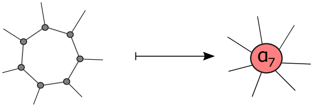

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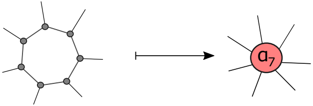

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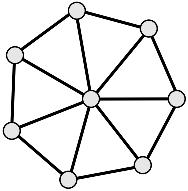

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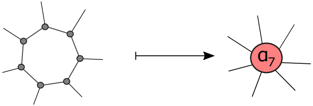

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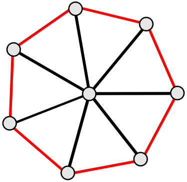

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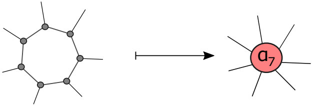

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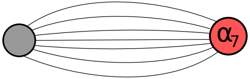

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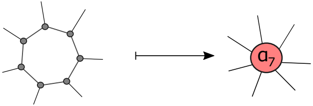

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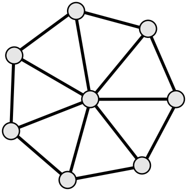

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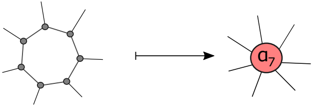

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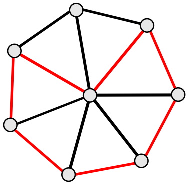

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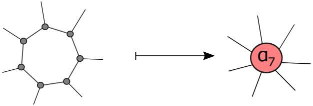

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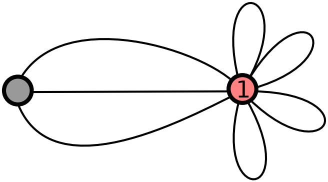

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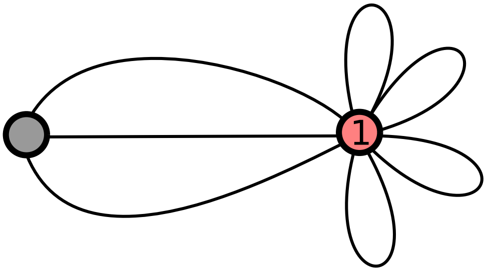

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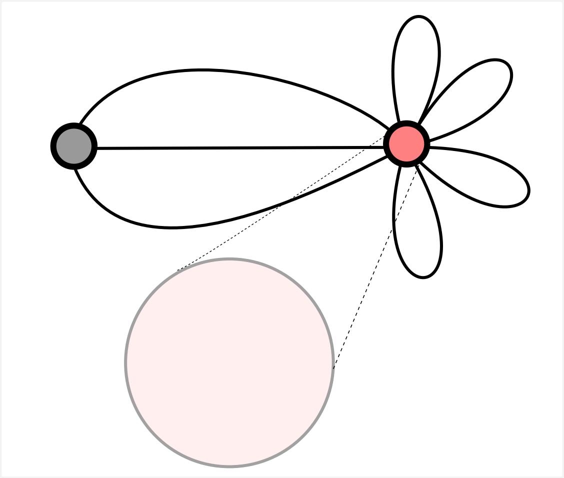

- Needs to be labeled by a class in $H_6(\Gamma_{1,11})$.

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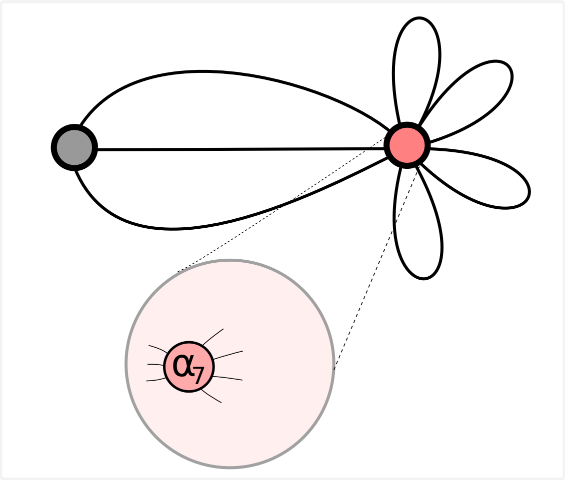

- Start $\alpha_7 \in H_6(\Gamma_{1,7})$.

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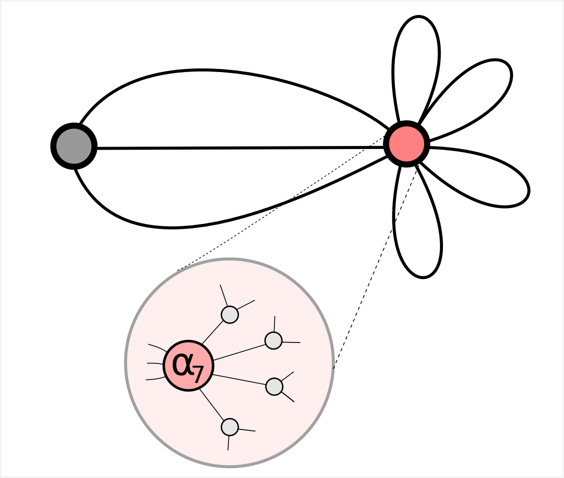

- Compose with a copy of $H_0(\Gamma_{0,3})$ for each tadpole.

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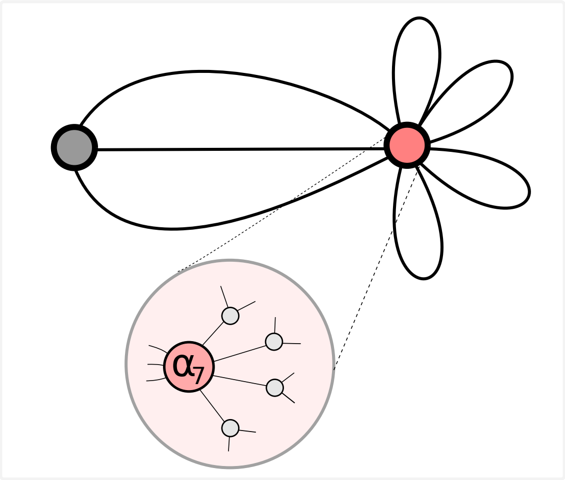

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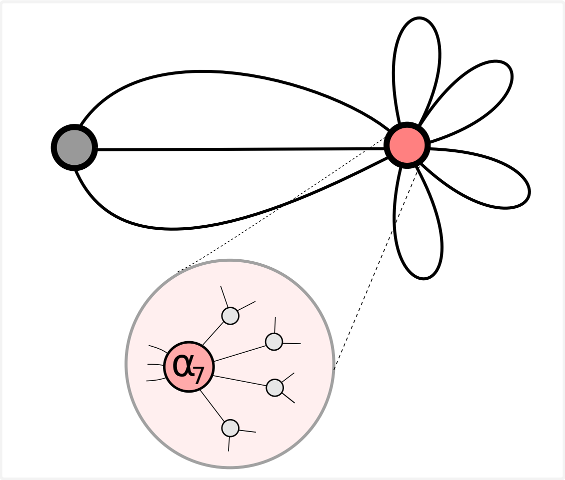

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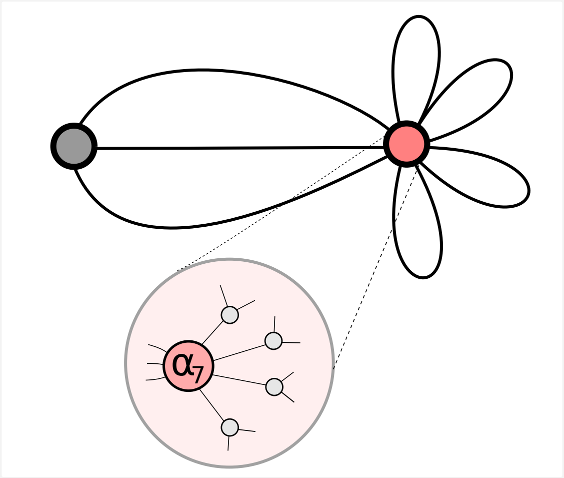

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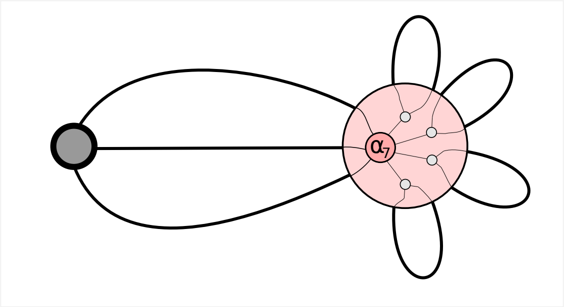

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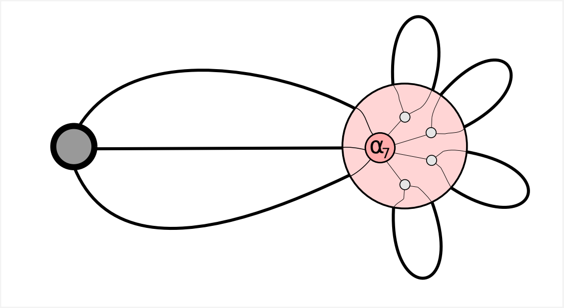

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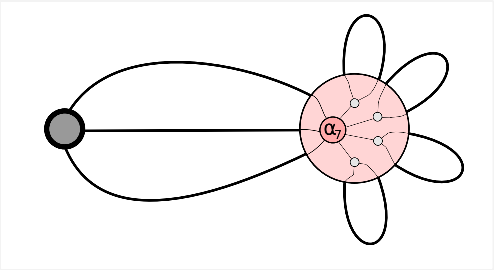

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With the expansion differential $[d(\theta_{2j+1})] = \sigma_{2j+1}$.

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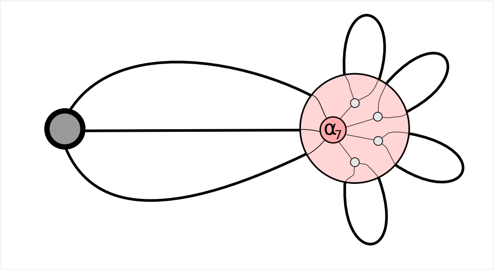

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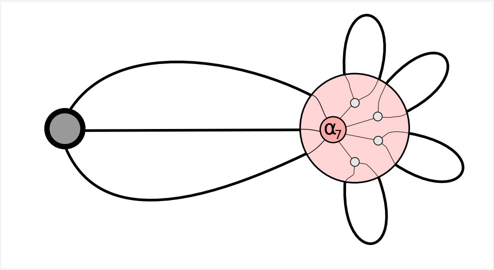

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With the expansion differential $[d(\theta_{2j+1})] = \sigma_{2j+1}$. *Dually, with contraction differential the wheel graph is not a boundary. I.e. the wheel graph represents a non-trivial class in* GC_2^*.

- This result is known, but here required no knowledge of $\mathfrak{grt}_1$.

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Bigger picture

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We can consider Massey products for any homology modular operad.

- Other flavors of graph homology.
  - Commutative in, Lie out.
  - Associative operad.
- $H_*(\mathcal{M}_{g,n}) \Rightarrow \mathrm{ft}(H(\mathcal{M}))(g,n) \sim H(\overline{\mathcal{M}}_{g,n})$.

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Questions? Answers?

`benward@bgsu.edu`

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Thank you!