3-manifolds with finite fund. group ffg

$S^3 = \{\underline{x} \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\} \subset \mathbb{R}^4$

$\pi_1 S^3 = 1$ simply-connectd

S^3 topological mfld

smooth mfld

Riemannian mfld with $K \equiv 1$

"Spherical Space Form" ssf

Our mflds will be closed $\left(= \begin{matrix}\text{compact} \\ \text{w/o } \partial\end{matrix}\right)$

$\mathbb{R}P^3 = S^3 / \underline{x} \sim -\underline{x}$, $\pi_1 = C_2$, ssf

<u>Def:</u> isometry is isomorphism $f : X \to Y$ of metric spaces

Poincaré Conj (Perelman's Thm)

A simply connected 3-manifold is homeo to S^3.

<u>Cor:</u> M^3 has ffg

$$\iff \exists \text{ cover } \begin{array}{c} S^3 \\ \downarrow \\ M^3 \end{array}$$

$\iff \exists$ finite $G \curvearrowright S^3$ freely

$$S^3/G \cong M^3$$

Thm (Perelman et al)

a) M^3 ffg $\iff$ M ssf

b) Let M & N be ssf.

$$M \underset{\text{homeo}}{\cong} N \iff M \underset{\text{isometric}}{\cong} N$$

Thus the classification of 3-manifolds with ffg is algebra!! (Representation Theory).

<u>Def:</u> Orthogonal Group

$$O(n) = \{A \in M_n\mathbb{R} \mid AA^t = I\}$$

$$= \{\text{isometries } f : \mathbb{R}^n \to \mathbb{R}^n \mid f(0) = 0\}$$

$$= \{\text{isometries } f : S^{n-1} \to S^{n-1}\}$$

Def: $G < O(n)$ is fixed point free (fpf) if $\forall g \neq 1, x \in S^{n-1}, gx \neq x$

$$\underbrace{\text{3-mflds with ffg}}_{\text{homeo}} \cong \underbrace{\text{finite fpf } G < O(4)}_{\text{conjugacy}}$$

Q1 (Enumeration)

Which groups are ffg of 3-mflds?

Q2 (Restriction)

Which groups are not ffg of 3-mflds

Q3 (Classification): Classify M^3 s.t.

$\pi_1 M^3 = G$.

$\pi_1 M^3 = 1 \Rightarrow M^3 = S^3$

$\pi_1 M^3 = C_2 \Rightarrow M^3 = \mathbb{R}P^3$

<u>Pf:</u> $r \in O(4)$, $r^2 = I$, $r \neq I$

r similar to $\begin{pmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \pm 1 & \\ & & & \pm 1 \end{pmatrix}$ ..

fpf
$\Rightarrow r$ similar to $-I$

$\Rightarrow r = -I$

Cor: $\nexists M^3$ s.t. $\pi_1 M^3 = C_2 \times C_2$

Rotation matrix $R(n,q) = \begin{pmatrix} \cos\frac{2\pi q}{n} & -\sin\frac{2\pi q}{n} \\ \sin\frac{2\pi q}{n} & \cos\frac{2\pi q}{n} \end{pmatrix}$

$r \in O(4)$, order n, fpf $\Rightarrow$

$r \sim \begin{pmatrix} R(n,q_1) & 0 \\ 0 & R(n,q_2) \end{pmatrix}$ $(q_i, n) = 1$

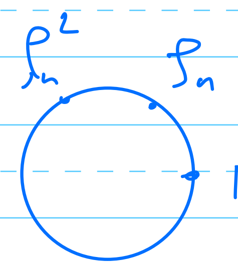
Cor: $G < O(4)$ fpf $\Rightarrow G < SO(4)$

$SO(4) = \{A \in O(4) \mid \det A = 1\}$.

Lens Spaces $L(n, q)$

$0 < q < n$, $(q, n) = 1$

cyclic $C_n = \langle r \rangle$



$$\rho_n = e^{2\pi i/n}$$

$$S^3 = \{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1 \}$$

$C_n \curvearrowright S^3$ via

$$r(z_1, z_2) = (\rho_n z_1, \rho_n^q z_2)$$

$$L(n, q) := S^3 / C_n = S^3 /_{\underline{z} \sim r\underline{z}}$$

Thm: a) $\pi_1 M^3 = C_n \Rightarrow M^3 \underset{\text{homeo}}{\cong} L(n, q)$

b) $L(n, q) \cong L(n', q') \Rightarrow n = n'$ & $q' \equiv \pm q^{\pm 1} \pmod{n}$

Quaternions $\mathbb{H}$

$x = a + bi + cj + dk$

$a, b, c, d \in \mathbb{R}$

$i^2 = -1,\ j^2 = -1,\ k = ij,\ ij = -ji \ \ldots$

$\bar{x} := a - bi - cj - dk$ conjugate.

$x\bar{x} = a^2 + b^2 + c^2 + d^2$

$\Rightarrow \mathbb{H}$ skew field $(\mathbb{H}^\times = \mathbb{H} - 0)$

$\vdots \quad S^3 = \{x \in \mathbb{H} \mid x\bar{x} = 1\} < \mathbb{H}^\times$ subgroup

$$\begin{aligned} S^3 &\longrightarrow \mathrm{Homeo}(S^3) \\ x &\longmapsto (y \mapsto xy) \end{aligned}$$

$S^3 \curvearrowright S^3$ fpf

$M^3 = S^3 / Q_8 \qquad Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} < S^3$

generalized quaternion group

$Q_{4n} = \langle e^{2\pi i/2n}, j \rangle < S^3$

Restrictions on ffg

$\cancel{C_2 \times C_2}$

p, q primes (maybe $p = q$)

Def: G satisfies pq-condition if all subgroups of order pq are cyclic

pq-Theorm: If $G < O(n)$ is fpf then G satisfies pq-conditions $\forall p, q$

Eg. $C_p \times C_p$, dihedral not $\pi_1 M$

Thm: Let G be a p-group satisfying p^2-condition. Then G is cyclic or Q_{2^n}

Variants of $O(n)$

Special Orthogonal $SO(n) = \{A \in O(n) \mid \det A = 1\}$

Unitary group $U(n) = \{A \in M_n\mathbb{C} \mid AA^* = I\}$

Symplectic group $Sp(n) = \{A \in M_n \mathbb{H} \mid AA^* = I\}$

$SO(n) \subset O(n)$

$U(n) \subset O(2n)$

$Sp(n) \subset U(2n)$; $S^3 = Sp(1)$

Two double covers

$$\pi_1 SO(n) = C_2 \qquad n>2$$

$$1 \to \{\pm 1\} \longrightarrow S^3 \longrightarrow SO(3) \longrightarrow 1$$

$$x \longmapsto \left(y = bi + cj + dk \mapsto x y \bar{x} \right)$$

$$1 \to \{\pm 1\} \longrightarrow S^3 \times S^3 \longrightarrow SO(4) \to 1$$

$$(x, y) \longmapsto \left(z \in \mathbb{H} \mapsto x z \bar{y} \right)$$

Want to find all $\pi_1 M^3$ finite

Step 1: Find all finite subgroups of $SO(3)$

Step 2: Use $S^3 \twoheadrightarrow SO(3)$ to find all finite subgroups of S^3

Step 3: Use Goursat's Lemma to find all finite subgroups of $S^3 \times S^3$

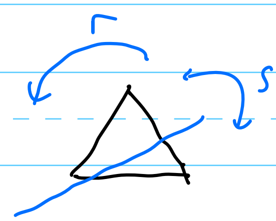
Step 4: Find all (fpf) finite subgroups of $SO(4)$

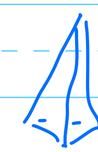
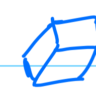
SO(3)

Thm (Klein) The finite subgroups of $SO(3)$ are

- cyclic C_n
- dihedral $D_{2n} = C_n \rtimes C_2$

$r \quad s \qquad srs^{-1} = r^{-1}$



- tetrahedral group $T = \mathrm{Isom}$ 
  order 12
- octahedral group $O = \mathrm{Isom}$ 
  order 24
- icosahedral group I
  order 60

Idea of proof: $G < SO(3)$

Case 1: G leaves plane invariant. $G \subset O(2)$

Case 2: Choose $x \in \mathbb{R}^3$, $x \neq 0$. Convex hull of Gx is regular solid.

$$1 \to \{\pm 1\} \to S^3 \xrightarrow{\pi} SO(3) \to 1$$

Cor: The finite subgroups of S^3 are

- cyclic
- binary dihedral $\pi^{-1} D_{2n} = Q_{4n}$ (= generalized quaternionic)
- $T^* = \pi^{-1} T$ binary tetrahedral
- $O^* = \pi^{-1} O$ " octahedral
- $I^3 = \pi^{-1} I$ " icosahedral

S^3 / I^* Poincaré homology sphere.

Goursat's Lemma gives a finite

subgroup of $S^3 \times S^3$

$$1 \to \{\pm 1\} \to S^3 \times S^3 \xrightarrow{\pi} SO(3) \to 1$$

Hopf's List: The ffg of 3-manifolds:

C_n, D^*_{4n}, T^*, O^*, I^*, $D'_{2^k(2n+1)}$ $(k \geq 2)$

$T'_{8 \cdot 3^k}$ and direct products of

these with C_ℓ where $(\ell, |G|) = 1$

$$D' = \langle x, y \mid x^{2^k} = 1 = y^{2n+1}, \ xyx^{-1} = y^{-1} \rangle$$

$$T' = \langle x, y, z \mid x^2 = (xy)^2 = y^2, \ zxz^{-1} = y^{-1}, \ zyz^{-1} = xy, \ z^{3^k} = 1 \rangle$$

References

Davis–Milgram: A Survey of the Spherical Space Form Problem

Orlik: Seifert Manifolds