Groups with periodic cohomology

Def: topological spherical space form

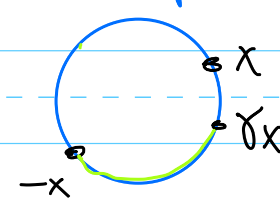
is a mfld covered by the sphere

i.e. $M = S^k / G$; finite $G \curvearrowright S^k$ freely

Q: What G occur? Classification?

$\gamma : S^k \to S^k$ has no fixed pts

$\Rightarrow \gamma \simeq$ antipodal



$\Rightarrow \deg \gamma = (-1)^{k+1}$

$\therefore$ k even $\Rightarrow \deg \gamma = -1 \Rightarrow G = 1$ or C_2

Thus assume k odd

Goal: G must satisfy p^2-condition $\forall p$

i.e. $\quad C_p \times C_p \not< G$

Cohomology of groups

$K(G,1) = BG$ is based CW cplx st.

$\pi_1 = G, \quad \pi_i = 0 \quad i \neq 1$

Remark: G finite $\Longrightarrow \tilde{H}_* BG$ finite

$\overset{UCT}{\Longrightarrow} \quad \tilde{H}_n(BG) = \tilde{H}^{n+1}(BG)$

Def 1: $H^n G = H^n(BG)$

$\mathbb{Z}G$ group ring
$\uparrow$
$n_1 g_1 + \cdots + n_\ell g_\ell$

Def 2: $H^n G = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$

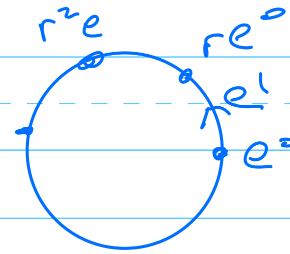
"free resolution of A" ($\leftarrow R$-module)

$F_A = \{ \cdots \to F_n \to \cdots \to F_1 \to F_0 \}$

$$H_i F_A = \begin{cases} A & i = 0 \\ 0 & i > 0 \end{cases}$$

$$\mathrm{Ext}^n_R(A,B) = H^n(\mathrm{Hom}_R(F_A, B))$$

Cyclic $C_h = \langle r \rangle \circlearrowright S^1 \quad rz = \zeta_h z$



$$C_\bullet S^1 \cong \mathbb{Z}C_h \xrightarrow{r-1} \mathbb{Z}C_h$$

$\Rightarrow$ "periodic free resolution"

$$0 \to \mathbb{Z} \xrightarrow{\eta} \mathbb{Z}C_h \xrightarrow{r-1} \mathbb{Z}C_h \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

$$1 \longmapsto N := \sum r^i \quad \text{"norm"}$$

Concatenate to get free resolution

$$\begin{tikzcd} F_{\mathbb{Z}} = \arrow[r, "r-1"] & \mathbb{Z}C_h \arrow[rr, "N"] \arrow[dr, "\varepsilon"'] & & \mathbb{Z}C_h \arrow[r, "r-1"] & \mathbb{Z}C_h \\ & & \mathbb{Z} \arrow[ur, "\eta"'] & & \end{tikzcd}$$

$$\mathrm{Hom}(F, \mathbb{Z}) = \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{h} \mathbb{Z} \xleftarrow{0} \mathbb{Z}$$

$$\widetilde{H}^n(C_h) = \begin{cases} \mathbb{Z}/h & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Theorem: If $G \curvearrowright S^{n-1}$ freely, n even

then $H^i G \cong H^{i+n} G \quad \forall i > 0$

$$H^n G = \mathbb{Z}/|G|$$

<u>Pf:</u> S^{n-1}/G CW-complex

$\rightsquigarrow$ periodic free resolution

$$0 \to \mathbb{Z} \to C_{n-1}(S^{n-1}) \to \cdots \to C_0(S^{n-1}) \to \mathbb{Z} \to 0$$

$$C_1 \to C_0 \to C_{n-1} \to \cdots \to C_0 \to \mathbb{Z} \to 0$$

$\Rightarrow \quad H^{i+n} G \cong H^i G$

S^{n-1}/G has single 0 & $n-1$-cell $\Rightarrow$ $C_0 \to C_{n-1}$ $\quad = \quad$ $\mathbb{Z}G \xrightarrow{N} \mathbb{Z}G$

QED

Another proof: Spectral Sequences

$$\begin{tikzcd} G \arrow[r] & S^{n-1} \arrow[d] \\ & S^{n-1}/G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r] & EG \arrow[d] \\ & BG \end{tikzcd}$$

$$\begin{tikzcd} EG \arrow[r] & S^{n-1} \times_G EG \arrow[d, "\simeq"] \\ & S^{n-1}/G \end{tikzcd} \qquad \begin{tikzcd} S^{n-1} \arrow[r] & EG \times_G S^{n-1} \simeq S^{n-1}/G \arrow[d] \\ & BG \end{tikzcd}$$

$$E_2^{p,q} = H^p(G; H^q(S^{n-1})) \Longrightarrow H^{p+q}(S^{n-1}/G)$$

$H^i(S^{n-1}/G) = 0 \quad i > n-1 \quad ; \quad H^{n-1}(S^{n-1}/G) \xrightarrow[\cdot |G|]{\cdot \deg} H^{n-1}S^{n-1}$

$$E^2 \qquad \begin{array}{c|ccccc} n-1 & H^0G & H^1G & & & \\ & & \ker = dH^0 & \cong & \cong & \\ 0 & H^0G & H^1G & H^nG & H^{n+1}G & \\ & & & \overset{\shortparallel}{\mathbb{Z}/d} & & \end{array}$$

$$E^\infty \qquad \begin{array}{c|cccccc} n-1 & dH^0 & 0 & 0 & \cdots & 0 \\ & & & & & \\ 0 & H^0G & H^1G & H^{n-1}G & 0 & \end{array}$$

G satisfies "pq-condition"
$\iff$ all subgroups of order pq are cyclic.

Theorem: If $G \curvearrowright S^{n-1}$ freely, n even

then $H^i G \cong H^{i+n} G \quad \forall i > 0$

$$H^n G = \mathbb{Z}/|G|$$

Cor: If $G \curvearrowright S^{n-1}$ freely, then

G satisfies p^2-condition ($C_p \times C_p \not\hookrightarrow G$)

<u>Pf</u>: Assume $C_p \times C_p \hookrightarrow G$.

$C_p \times C_p \curvearrowright S^{n-1}$ freely

$\Rightarrow H^* C_p \times C_p$ is n-periodic

Contradicts Künneth Thm.

Def: G has period n if $\hat{H}^n G = \mathbb{Z}/|G|$

Period of G is smallest n.

E.g. $\text{Period}(C_n) = 2$ $\quad$ $\text{Period}(Q_8) = 4$

Spectral Sequence exercise

p odd $\Rightarrow$ $\text{Period}(D_{2p} = C_p \rtimes_{-1} C_2) = 4$

$\text{Period}(C_7 \rtimes_2 C_3) = 6$

Thm (Cartan-Eilenberg)

TFAE

a) G has period n some n

b) G satisfies p^2-condition $\forall p$.

c) $G_p = p$-sylow subgroup $\quad$ p odd $\Rightarrow G_p$ cyclic

$p = 2 \Rightarrow G_2$ cyclic or generalized quaternionic

Thm (Cartan-Eilenberg, Swan)

TFAE

a) G has period n $\quad (\hat{H}^n G = \mathbb{Z}/|G|)$

b) $\exists i, \forall M, \ H^i(G;M) \cong H^{i+n}(G;M)$

c) $\forall i > 0, \forall M, \ H^i(G;M) \cong H^{i+n}(G;M)$

d) $\exists$ periodic free res $0 \to \mathbb{Z} \to F_{n-1} \to \cdots \to F_0 \to \mathbb{Z} \to 0$

e) $\exists$ CW X, $\dim X = n-1$, $\tilde{X} \simeq S^{n-1}$, $\pi_1 X = G$

f) $\exists$ periodic proj res

$$0 \to \mathbb{Z} \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$$

P_i f.g. proj $\mathbb{Z}G$-modules

Variants of Spherical Space Form Problem

G = finite group with period n

Q1: $\exists$? Riem mfld M^{n-1} with $K \equiv 1$

& $\pi_1 M = G$?

Q2: $\exists$? mfld M with $\pi_1 M = G$ & $\tilde{M} \cong S^{n-1}$

Q3: $\exists$? finite CW complex X^{n-1}

with $\pi_1 X = G$. $\tilde{X} \simeq S^{n-1}$

Q4: $\exists$? CW complex X^{n-1} with $\pi_1 X = G$

$\tilde{X} \simeq S^{n-1}$

Q1: $\exists$? Riem mfld M with $K \equiv 1$

& $\pi_1 M = G$?

$\iff$ $G < O(n)$ fpf

Solved: ($\exists$ list)

Dim 3 Hopf

Dim > 3 Vincent / Wolf

$G < O(n)$ fpf $\Rightarrow$ G satisfies

pq-condition

Q2: $\exists$? mfld M with $\pi_1 M = G$ & $\tilde{M} \cong S^{n-1}$

Thm (Petrie) $\exists M^5$, $\tilde{M} \cong S^5$ s.t.

$\pi_1 M = \mathbb{Z}_7 \rtimes_2 \mathbb{Z}_3$

Pf: "surgery on Brieskorn mfld"

$\pi_1 M^5$ does not satisfies $3 \cdot 7$-condition
hence not space form

Thm (Milnor; Lee; Davis) dihedral

group D_{2p} does not act freely on sphere

Thm (Madsen–Thomas–Wall)

G acts freely on S^k, some k

$\iff$ G satisfies p^2 and $2p$-conditions

Q4: $\exists$? CW complex X^{n-1} with $\pi_1 X = G$

$\tilde{X} \simeq S^{n-1}$

"Hurewicz"

$\Longleftrightarrow$ - $0 \to \mathbb{Z} \to F_{n-1} \to \cdots \to F_0 \to \mathbb{Z} \to 0$

Swan

$\Longleftrightarrow$ $H^n G = \mathbb{Z}/|G|$

Swan

$\Longleftrightarrow$ $\exists$ $0 \to \mathbb{Z} \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$

f.g proj

$\forall G$, $\exists X$ s.t. $\pi_1 X = G$, $\tilde{X} \simeq S^{n-1}$

$X = S^{n-1} \times K(G, 1)$

Q3: $\exists$? finite CW complex X^{n-1}

with $\pi_1 X = G$.

"Hurewicz"

$\Longleftrightarrow$ $\exists$ $0 \to \mathbb{Z} \to F_{n-1} \to \cdots \to F_0 \to \mathbb{Z} \to 0$

f.g. free.

Eg (Swan) Yes for $G = D_{2p}$

E.g. (Davis) No. $G = Q_{16} \rtimes C_3$

$= \langle x, y, z \mid x^4 = y^2, x^8 = 1, yxy^{-1} = x^{-1}, xzx^{-1} = z^{-1}, yz = zy \rangle$

Then $G = \pi_1 X^3$, $\tilde{X}^3 \simeq S^3$

but there $\nexists$ finite such X.

i.e. $\exists$ f.g. proj $0 \to \mathbb{Z} \to P_3 \to \cdots \to P_0 \to \mathbb{Z} \to 0$

but no f.g free $0 \to \mathbb{Z} \to F_3 \longrightarrow \cdots \to F_0 \to \mathbb{Z} \to 0$

$$\underbrace{\sum (-1)^i [P_i]}_{\neq 0} \in \underbrace{\tilde{K}_0(\mathbb{Z}G)}_{T}$$