Talk 1: closed 3-mfld with finite π_1 = spherical space form of type $(G,3)$:= closed M^3 with $\pi_1 M = G$ & $\tilde{M} = S^3$

Talk 3: finite Swan complex of type $(G,3)$ = dim 3 finite CW complex X with $\pi_1 X = G$ & $\tilde{X} \simeq S^3$

Q: What G occur? What homotopy types?

M ssf of type (G,d)

$\Rightarrow \exists$ Swan complex of type (G,d)

CW X s.t. $\pi_1 X = G$, $\tilde{X} \simeq S^d$

Pf "triangulate M"

$\Leftrightarrow \exists$ $\mathbb{Z}G$-resolution of period $d+1$

$$0 \to \mathbb{Z} \to F_d \to \cdots \to F_0 \to \mathbb{Z} \to 0$$

free $/\mathbb{Z}G$

Pf: $F_i = C_i(\tilde{X})$

$\Leftrightarrow$ G has cohomology of period $d+1$

$$H^{d+1}(G) = \mathbb{Z}/|G|$$

Pf: Find chain homotopic complex

$$0 \to \mathbb{Z} \to \mathbb{Z}G \to F_{d-1}' \to \cdots \to F_1' \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

Thm (Swan/Wall) G finite

TFAE

a) $\exists$ Swan complex type (G, d)

b) $\exists$ $\mathbb{Z}G$-resolution of period $d+1$

c) $H^{d+1} G \cong \mathbb{Z}/|G|$ G has coh of period $d+1$

Pf: $a \Rightarrow b$ $C_\cdot(\tilde{X})$

$b \Rightarrow a$ "Hurewicz"

$b \Rightarrow c$ ✓   che

$c \Rightarrow b$ homological algebra Swan / wall

$\forall$ space X, first k-invariant

$k(X) \in H^{d+1}(\pi_1 X; \pi_d X)$ where

$\pi_i X = 0$ for $1 < i < d$

Geom Def: $k(X)$ is first obstruction to extending $K(\pi_1 X, 1)^d \to X$ to $K(\pi_1 X, 1)$

Alg Def: Let $P_\cdot$ be $\mathbb{Z}\pi_1 X$-res of $\mathbb{Z}$

$$\begin{tikzcd}
 & \arrow[r] & P_{d+1} \arrow[r] \arrow[d, dashed] & P_d \arrow[r] \arrow[d, dashed] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, dashed] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[rr] & & Z_d \tilde{X} \arrow[r] \arrow[d] & C_d \tilde{X} \arrow[r] & \cdots \arrow[r] & C_0 \tilde{X} \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\
 & & H_d \tilde{X} = \pi_d X & & & & &
\end{tikzcd}$$

$k(X) = [P_{d+1} \to \pi_d X] \in H^{d+1}(\pi_1 X; \pi_d X)$

Swan complex is polarized:

fixed iso $\pi_1 X \cong G$

fixed hty class $\tilde{X} \simeq S^d$

i.e. X is oriented & has $\pi_1 = G$

Homotopy types of Swan complexes (and periodic resolutions)

Thm: a) k is bijection

$$k: \begin{array}{c}\text{polarized}\\ \text{htpy types}\\ \text{of } (G,d) \text{ Swan}\\ \text{complexes}\end{array} \xrightarrow{\sim} \begin{array}{c}\text{additive}\\ \text{generators}\\ \text{of } \mathbb{Z}/|G|\end{array}$$

$$\cap$$

$$H^{d+1}(G;\mathbb{Z})$$

$$\uparrow$$

$$\pi_d$$

b) X, Y (G,d) S.C.

$\ell k(X) = k(Y) \iff \exists \deg \ell$ map $X \to Y$ identity on π_1

$(\ell, |G|) = 1$

Cor: homotopy types of (G,d) Swan complexes are given by $\left((H^{d+1}G)^{\times}/\pm 1\right)/\mathrm{Aut}\, G$

Examples: Len spaces & S^3/Q_8

All k-invariants are realized by lens spaces $k(L(p,q)) = [q] \in \mathbb{Z}/p = H^4(C_p)$

$L(p,q) \simeq L(p,q') \iff q \equiv \pm k^2 q' \pmod{p}$

e.g. $L(7,1) \not\simeq L(7,2)$

$\left(H^4(Q_8)^\times/\pm 1\right)/\mathrm{Aut}\, Q_8$

$= H^4(Q_8)^\times/\pm 1 = (\mathbb{Z}/8)^\times/\pm 1 = \langle 1, 3\rangle$

One k-invariant represented by ssf S^3/Q_8

$\exists$? finite $(Q_8, 3)$ Swan complex s.t. $k(X) = 3$? Answer: No

Projective Modules

Def: R-module P is projective if $\exists Q$ s.t. $P \oplus Q$ is free

Lemma TFAE

a) P projective

b) Every surjection to P splits

$$\begin{tikzcd} N \arrow[r, two heads, shift right] & P \arrow[l, dashed, bend right] \end{tikzcd}$$

c) Every SES $0 \to M \to N \to P \to 0$ splits

d) P = image of projection

$\exists \pi : F \to F$ s.t. $\pi \circ \pi = \pi$, $P \cong \pi(F)$.

e)
$$\begin{tikzcd} & P \arrow[d] \arrow[dl, dashed] & \\ M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

Def: A projective resolution of period $d+1$ is an exact sequence

$$0 \to \mathbb{Z} \to P_d \to \cdots \to P_0 \to \mathbb{Z} \to 0$$

where P_i are f.g. proj $\mathbb{Z}G$-mods

k-invariant $k(P_\cdot) \in H^{d+1} G$

Thm (Swan/Wall)

Let G be period $d+1$

a) All k-invariants are realized by periodic proj res $k(P_\cdot) \in (H^{d+1} G)^\times$

b) $k(P_\cdot) = k(Q_\cdot)$ iff $P_\cdot \underset{che}{\simeq} Q_\cdot$

c) $k(P_\cdot) = k(\text{finite s.c.}) \iff \chi(P_\cdot) = 0$
$\in \tilde{K}_0(\mathbb{Z}G)$

Algebraic K-theory

$K_0 R = Gr(\text{f.g. proj}, \oplus)$

$\tilde{K}_0 R = K_0 R / \text{free modules}$

e.g. $K_0 \mathbb{R} \cong \mathbb{Z}$ $\quad \tilde{K}_0(\mathbb{R}) = 0$

$K_0 M_2 \mathbb{R} \cong \mathbb{Z}$ $\quad \tilde{K}_0(M_2(\mathbb{R})) = \mathbb{Z}/2$

$\mathbb{R}^2 \longmapsto 1$

Swan module: $(\ell, |G|) = 1$

$P_\ell := \ker(\mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z}/\ell)$ is proj

Swan Subgroup $T(G) = \{[P_\ell] \in \tilde{K}_0(\mathbb{Z}G)\}$

e.g. $T(Q_8) = \{\mathbb{Z}G, P_3\}$

Thm (Swan) $\ell k(P_\cdot) = k(Q_\cdot)$

$\Rightarrow \chi(P_\cdot) = \chi(Q_\cdot) + [P_\ell] \in \tilde{K}_0(\mathbb{Z}G)$

Cor: $\exists$ $(Q_8, 3)$ Swan complex not h.e. to finite complex.

Finiteness Obstructions

Hopf's List $\subset$ List of period 4 groups

ssf $\qquad\qquad$ sc

Def: X is $(G,3)$ s.c.

Wall finiteness obstruction

$[X] = \chi(P_\cdot) \in \tilde{K}_0 \mathbb{Z}G$.

where $P_\cdot(X) := P_\cdot(P_\cdot)$

Fact: $[X] = 0 \iff X \simeq$ finite CW

Def: Swan finiteness obstruction

for period 4 group

$\sigma_4 G := \chi(P_\cdot) \in \tilde{K}_0(\mathbb{Z}G)/T$

$\sigma_4 G = 0 \iff \exists$ finite $(G,3)$ s.c.

Thm (Milgram) $\exists$ period 4 G s.t. $\sigma_4 G \neq 0$

Thm (Davis) $G = C_3 \rtimes Q_{16}$

$\sigma_4 G \neq 0$ & G smallest possible group

i.e. $\exists X^3 \quad \pi_1 X^3 = G, \; \tilde{X}^3 \simeq S^3$

& $\exists \; 0 \to \mathbb{Z} \to P_3 \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0$

$\nexists$ finite Y^3 s.t. $\pi_1 Y = G, \; \tilde{Y} \simeq S^3$

& $\nexists \quad 0 \to \mathbb{Z} \to F_3 \to F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$

f.g. free

Program of Davis / Nicholson

Evaluate all Wall & Swan

finiteness obstructions for

period 4 groups