Introduction to Diffeology

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I am Emilio Minichiello, a math PhD student at the CUNY Graduate Center in New York City. I study the intersection of (higher) category theory and differential geometry. Diffeology sits snugly in this intersection.

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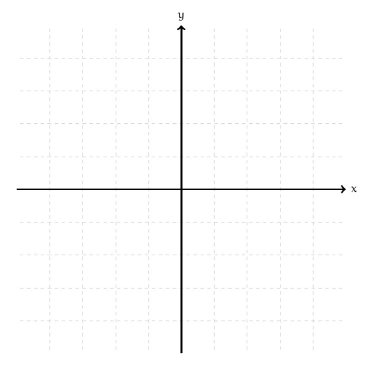
Introduction

If you don't know much category theory, you can think of it this way. Consider the union of the x-axis and the y-axis in $\mathbb{R}^2$, we will call it Axes.



This is the result of gluing two copies of $\mathbb{R}$ together at the origin. It isn't a manifold!

Introduction

For last 200 years or so, an astonishing amount of theory has been developed for good ol' finite dimensional smooth manifolds. These involve elaborate constructions and powerful theorems. But none of them apply to our friend Axes!

So here's the idea:

1. Find a category $\mathcal{C}$ such that the category Man embeds into it, Man $\hookrightarrow \mathcal{C}$, and
2. We see how much of our classical theory extends to this new category.

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Diffeological Spaces

Diffeological Spaces

The category Diff of diffeological spaces is one such choice for $\mathcal{C}$.

We say a manifold U is a **cartesian space** if $U \cong \mathbb{R}^n$ for some $n \in \mathbb{N}$. If X is a set, we call a set function $p : U \to X$ a **parametrization** if U is cartesian.

We say a collection of subsets $\mathcal{U} = \{U_i \subseteq U\}$ where each U_i is cartesian is an **open cover** if $\bigcup_i U_i = U$.

Diffeological Spaces

A **diffeological space** consists of a set X, and a collection $\mathcal{D}$ of parametrizations $p : U \to X$ satisfying the following three axioms:

1. $\mathcal{D}$ contains all the points $\mathbb{R}^0 \to X$,
2. If $p : U \to X$ belongs to $\mathcal{D}$, and $f : V \to U$ is a smooth map, then $pf : V \to X$ belongs to $\mathcal{D}$, and
3. If $\{U_i \subseteq U\}_{i \in I}$ is an open cover of U, and $p : U \to X$ is a parametrization such that $p|_{U_i} : U_i \to X$ belongs to $\mathcal{D}$ for every $i \in I$, then $p \in \mathcal{D}$.

If $(X, \mathcal{D})$ is a diffeological space, then we call elements $p \in \mathcal{D}$ **plots**.

Diffeological Spaces

Similarly to topology, any set S can be given two canonical diffeologies, the **discrete** $\mathcal{D}_\circ$ and **indiscrete** $\mathcal{D}_\bullet$ diffeologies. $\mathcal{D}_\circ$ consists of all the constant maps $U \to \mathbb{R}^0 \to S$, while $\mathcal{D}_\bullet$ consists of all parametrizations $U \to S$.

If M is a finite dimensional smooth manifold, then $\mathcal{D}_M = \{p : U \to M \;:\; p \text{ is smooth in the classical sense}\}$ forms a diffeology.

Thus every manifold is a diffeological space.

Diffeological Spaces

Let $(X, \mathcal{D}_X)$ and $(Y, \mathcal{D}_Y)$ denote diffeological spaces. We say a set function $f : X \to Y$ is **smooth** if for every $U \xrightarrow{p} X \in \mathcal{D}_X$, the composition $U \xrightarrow{p} X \xrightarrow{f} Y \in \mathcal{D}_Y$.

Proposition ([Igl13, Article 4.3]): If M and N are finite dimensional smooth manifolds, then a function $f : M \to N$ is smooth in the classical sense iff it is smooth as a map between diffeological spaces.

If we let Diff denote the category whose objects are diffeological spaces and whose morphisms are smooth maps, then the above Proposition tells us that the inclusion functor Man $\hookrightarrow$ Diff is fully faithful.

Now here are some ways in which diffeological spaces are very different from manifolds.

Let $(X, \mathcal{D})$ be a diffeological space and $A \subseteq X$ a subset. Then A inherits a canonical diffeology $\mathcal{D}_{\subseteq}$ defined as follows. A parametrization $p : U \to A$ is said to be a plot if the composition $U \xrightarrow{p} A \hookrightarrow X$ is a plot of X. We call $\mathcal{D}_{\subseteq}$ the **subset diffeology** on A.

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Diffeological Spaces

Similarly, let $(X, \mathcal{D})$ be a diffeological space and let $\sim$ be an equivalence relation on X. Then the quotient $X/{\sim}$ inherits a diffeology $\mathcal{D}_\sim$ defined as follows. A parametrization $p : U \to X/{\sim}$ is said to be a plot if there exists an open cover $\{U_i \subseteq U\}_{i \in I}$ and plots $p_i : U_i \to X$ such that the following diagram commutes

$$\begin{array}{ccc} U_i & \xrightarrow{p_i} & X \\ \big\downarrow & & \big\downarrow \pi \\ U & \xrightarrow[p]{} & X/{\sim} \end{array}$$

We call $\mathcal{D}_\sim$ the **quotient diffeology** on $X/{\sim}$.

Diffeological Spaces

Now let $(X, \mathcal{D}_X)$ and $(Y, \mathcal{D}_Y)$ be diffeological spaces, and consider the set $C^\infty(X, Y)$ of smooth maps $f : X \to Y$. This inherits a diffeology $\mathcal{D}_\to$, called the **functional diffeology**, defined as follows. A parametrization $p : U \to C^\infty(X, Y)$ is said to be a plot if the corresponding map $p^\# : U \times X \to Y$, defined by $p^\#(u, x) = p(u)(x)$, is smooth.

Thus diffeological spaces are closed under

1. subsets,
2. quotients, and
3. mapping spaces.

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Constructions on Diffeological Spaces

So we have a category $\mathcal{C} = \text{Diff}$, and it turns out to be an extremely nice category. It has all limits, colimits, and is cartesian closed. It also receives an embedding from the category of manifolds.

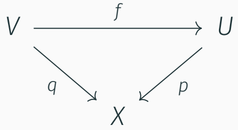
But what classical constructions can we extend to such a vast generalization?

Topology: Every diffeological space X can be given a canonical topology, called the D-topology, as follows. Say a subset $A \subseteq X$ is **D-open** if for every plot $p : U \to X$, the inverse image $p^{-1}(A)$ is open in U.

This defines a topology on X. The collection of topological spaces that arise from the D-topology of a diffeological space are called **Delta generated topological spaces** and they have very interesting properties, like being locally path connected. See [SYH10] and [DW14].

Constructions on Diffeological Spaces

Differential forms: If X is a diffeological space, a differential form ω on X consists of a collection of differential forms ω_p for every plot $p : U \to X$, with $\omega_p \in \Omega^k(U)$, such that if $f : V \to U$ is a map of plots, i.e. a smooth map such that the following diagram commutes:



then $f^*\omega_p = \omega_{pf} = \omega_q$.

deRham cohomology: With this definition, one can define the deRham cohomology ring $H^*_{dR}(X)$ for any diffeological space.

Here's an interesting diffeological space with nontrivial deRham cohomology. Let α be an irrational number, and consider the group $\mathbb{Z} + \alpha\mathbb{Z}$, of integers of the form $n + \alpha m$. This is a subgroup of $\mathbb{R}$, but notice that it is not a Lie subgroup, as it is dense in $\mathbb{R}$.

They are both naturally diffeological spaces. So we can take its quotient and get another diffeological space.

Constructions on Diffeological Spaces

Taking the quotient $T_\alpha := \mathbb{R}/(\mathbb{Z} + \alpha\mathbb{Z})$, we get a diffeological space, called the **irrational torus**. Its D-topology is coarse, i.e. is $\{\varnothing, T_\alpha\}$. This means that topology will not give interesting information for T_α.

However the irrational torus is nontrivial in diffeology, and that is reflected in its deRham cohomology:

$$H^0_{dR}(T_\alpha) = \mathbb{R}, \qquad H^1_{dR}(T_\alpha) = \mathbb{R}, \qquad H^k_{dR}(T_\alpha) = 0 \text{ if } k > 1.$$

The following notions have also been defined in diffeology and detailed extensively in [Igl13]:

1. Diffeological covering spaces,
2. Diffeological universal covering spaces,
3. Diffeological fiber bundles and principal bundles,
4. Diffeological homotopy groups.

Further every diffeological fiber bundle induces an exact sequence of diffeological homotopy groups. This allows us to compute the diffeological homotopy groups of the irrational torus:

$$\pi_0(T_\alpha) = 1, \qquad \pi_1(T_\alpha) = \mathbb{Z} + \alpha\mathbb{Z} \cong \mathbb{Z}^2, \qquad \pi_k(T_\alpha) = 0 \text{ if } k > 1.$$

Failure of Classical Results

Failure of Classical Results

Let X be a diffeological space and $x \in X$. Consider the category $\mathrm{Plot}_0(X, x)$ whose objects are smooth maps $p : U \to X$, where U is an open connected subset of $\mathbb{R}^n$ that includes the origin, and such that $p(0) = x$, and whose morphisms from $p : U \to X$ to $q : V \to X$ are smooth maps $f : U \to V$ such that $f(0) = 0$ and such that $qf = p$.

Then there is a functor $T : \mathrm{Plot}_0(X, x) \to \mathrm{Vect}$, given by $(p : U \to X) \mapsto T_0U$, where T_0U is the classical tangent space of U at 0. There is also a functor $\pi : \mathrm{Plot}_0(X, x) \mapsto \mathrm{Diff}_*$ to the category of pointed diffeological spaces, given by $(p : U \to X) \mapsto (U, 0)$.

Failure of Classical Results

We can take the left Kan extension

$$\begin{array}{ccc} \mathrm{Plot}_0(X,x) & \xrightarrow{T} & \mathrm{Vect} \\ {\scriptstyle \pi}\downarrow & \nearrow_{\mathrm{Lan}_\pi T} & \\ \mathrm{Diff}_* & & \end{array}$$

and this defines the **internal tangent space** of X at x:
$T_x X := (\mathrm{Lan}_\pi T)(X, x)$.

This agrees with the usual tangent space when X is a manifold. It is very reminiscent of the classical definition of the tangent space as an equivalence class of curves at the point x, but is defined for all diffeological spaces. This was first developed in [Hec95] and studied intensively in [CW15].

Failure of Classical Results

There is also a notion of external tangent space $\hat{T}_x X$, defined in [CW15]. It is the vector space of smooth derivations on germs of smooth functions from X to $\mathbb{R}$.

Classical differential geometry says that these two notions of tangent spaces at a point, as equivalence classes of curves, and as derivations of germs of smooth functions, are isomorphic on smooth manifolds.

This is not true in diffeology! There are diffeological spaces where $T_x X \neq \hat{T}_x X$.

Failure of Classical Results

The irrational torus T_α is an example.

From [CW15]: At any point $x \in T_\alpha$, $T_x(T_\alpha) \cong \mathbb{R}$ and $\hat{T}_x(T_\alpha) \cong \mathbb{R}^0$.

Failure of Classical Results

Another sacred result of classical differential geometry is the isomorphism between Čech cohomology $\check{H}^k(X, \mathbb{R})$ and deRham cohomology $H^k_{dR}(X)$ on smooth manifolds.

In [Igl20] it is proved that this is not the case for diffeological spaces! Indeed, the irrational torus has $\check{H}^1(T_\alpha, \mathbb{R}) = \mathbb{R}^2$ but $H^1_{dR}(T_\alpha) = \mathbb{R}$.

In fact more is true, one can pick out the actual obstruction to these groups being isomorphic!

Failure of Classical Results

Theorem ([Igl20]): For any diffeological space X, there is an exact sequence

$$0 \to H^1_{dR}(X) \to \check{H}^1(X, \mathbb{R}) \to \mathrm{Fl}^\bullet(X) \xrightarrow{c_1} H^2_{dR}(X) \to \check{H}^2(X, \mathbb{R})$$

where $\mathrm{Fl}^\bullet(X)$ denotes the group of diffeological $(\mathbb{R}, +)$-principal bundles that admit a connection 1-form ω.

The map c_1 is the curvature of the connection ω, and if the map c_1 is zero, then $\check{H}^1(X, \mathbb{R}) \cong H^1_{dR}(X)$.

If X is a manifold, then all $(\mathbb{R}, +)$-principal bundles will be trivial over X, and thus the deRham isomorphism in degree 1 holds for manifolds. However the irrational torus T_α has nontrivial principal $(\mathbb{R}, +)$-bundles.

Connection with Higher Topos Theory

Connection with Higher Topos Theory

In [Min22], diffeological spaces are thought of as certain kinds of higher sheaves, called simplicial presheaves, which are functors of the form $\mathcal{C}^{op} \to \mathrm{sSet}$.

This follows from the theorem of [BH09] that states that the category of diffeological spaces is equivalent to the category of concrete sheaves over the site of open subsets of cartesian spaces.

$$\mathrm{Diff} \simeq \mathrm{ConSh}(\mathrm{Open}).$$

Connection with Higher Topos Theory

The study of higher sheaves, also known as higher topos theory, blends classical sheaf theory, category theory, and homotopy theory together.

Higher topos theory is a very powerful framework to work with for differential geometry, and allows one to define and manipulate exotic cohomology theories. It also comes fully packed with a notion of **principal ∞-bundle**, which determines **nonabelian cohomology**.

Connection with Higher Topos Theory

Using this technology, one of the main results of [Min22] shows that for a diffeological group G diffeological principal G-bundles as defined in [Igl13] are actually examples of principal infinity bundles

$$\mathbb{R}\mathrm{Hom}(X, \mathbf{B}G) \simeq N\mathrm{DiffPrinc}_G(X).$$

This automatically defines another nonabelian cohomology group $\check{H}^1_\infty(X, G)$ closely related with the nonabelian Čech cohomology groups of [KWW21] and [Igl20].

Connection with Higher Topos Theory

I believe this bridge between diffeology and higher topos theory will be fruitful and allow us to port over more interesting and complicated objects into diffeology, like bundle gerbes, in a structured way.

Thank you!

Questions, comments? Feel free to email me at
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These slides are available at www.emiliominichiello.com.

References

[BH09] John C. Baez and Alexander E. Hoffnung. *Convenient Categories of Smooth Spaces*. 2009. arXiv: 0807.1704 [math.DG].

[CW15] J. Daniel Christensen and Enxin Wu. *Tangent spaces and tangent bundles for diffeological spaces*. 2015. arXiv: 1411.5425 [math.DG].

[DW14] Gordon Sinnamon Daniel Christensen and Enxin Wu. “The *D*-topology for diffeological spaces”. *Pacific Journal of Mathematics* 272.1 (Oct. 2014), pp. 87–110. URL: https://doi.org/10.2140%5C%2Fpjm.2014.272.87.

[Hec95] Gilbert Hector. “Géométrie et topologie des espaces difféologiques”. *Analysis and geometry in foliated manifolds (Santiago de Compostela, 1994)* (1995), pp. 55–80.

[Igl13] Patrick Iglesias-Zemmour. *Diffeology*. Vol. 185. American Mathematical Soc., 2013.

[Igl20] Patrick Iglesias-Zemmour. “Čech-de-Rham Bicomplex in Diffeology”. (2020).

[KWW21] Derek Krepski, Jordan Watts, and Seth Wolbert. *Sheaves, principal bundles, and Čech cohomology for diffeological spaces*. 2021. arXiv: `2111.01032 [math.DG]`.

[Min22] Emilio Minichiello. *Diffeological Principal Bundles and Principal Infinity Bundles*. 2022. arXiv: 2202.11023 [math.DG].

[SYH10] K. Shimakawa, K. Yoshida, and T. Haraguchi. *Homology and cohomology via enriched bifunctors*. 2010. arXiv: 1010.3336 [math.AT].