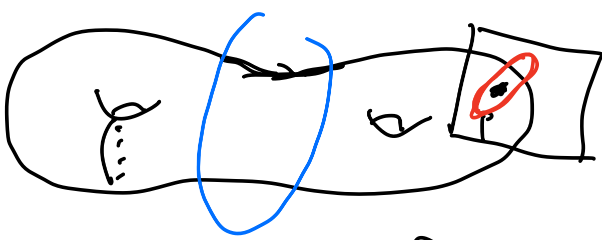
2/7/2023

IISER, Kolkata g&t seminar.

From geometry to dynamics and back

Marked length spectrum rigidity.

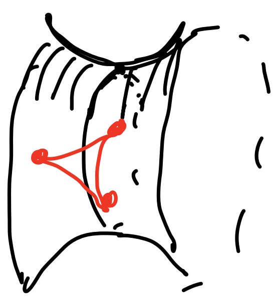
(S, g) g - Riemannian metric, negatively curved



$T_p^1 S = \{ v : \|v\|_p = 1 \}$

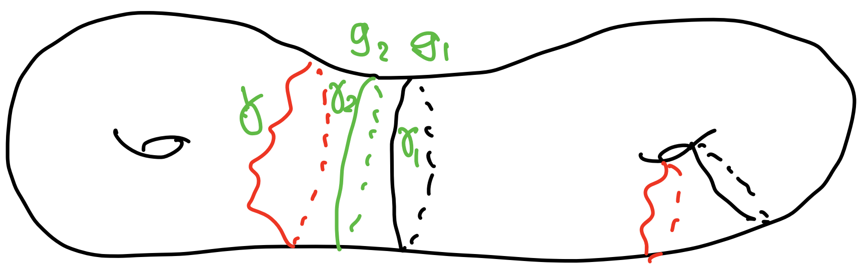
$T^1 S = \bigcup_{p \in S} T_p^1 S$

$\subset \mathbb{R}^3$



(S, g_1)

(S, g_2)



If γ is a loop let γ_i be the g_i-geodesic.

$l_{g_i}(\gamma_i)$ - length of γ_i w.r.t. to g_i.

0 ———————— ∞

Q: g_1, g_2 $\forall \gamma$ assume $\ell_{g_1}(\gamma_1) = \ell_{g_2}(\gamma_2)$.
When is this possible?

Yes. $\mathrm{Diff}(S)$ acting $\mathrm{Met}^{<0}(S)$

f a diffeo g a metric

$$f^*g(v_p, w_p) = g(Df\, v_p, Df\, w_p)$$

80

MLS rigidity: (Burns-Katok)

$(M, g_1), (M, g_2)$ neg. curv. compact manifold

$\forall \gamma \quad \ell(\gamma_1) = \ell(\gamma_2) \quad (*)$

Then $\exists f \in \mathrm{Diffeo}(M)$ s.t. $f^* g_2 = g_1$

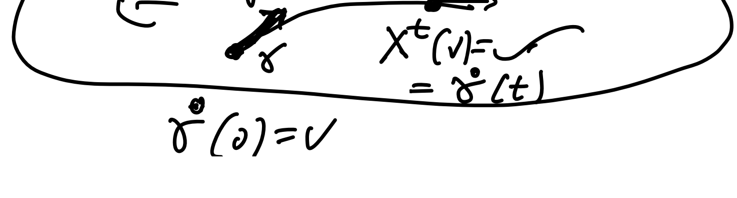
$f : (M, g_1) \to (M, g_2)$ is isometry.

Croke-Otal '90: true if $\dim M = 2$.

STEP 1: $(S, g_1), (S, g_2)$

$\rightsquigarrow$ geodesic flows

$X_1^t : T^1S \to T^1S$, $X_2^t : T^1S \to T^1S$



$\dot\gamma(0) = v$

neg. curv.

Easy: Assumption $(*)$ implies that geodesic flows are conjugate (Livshitz)

$\Downarrow$ Anosov property needed for step 1

$H : T^1S \to T^1S$ homeo

$$\boxed{H \circ X_1^t = X_2^t \circ H} \quad H \in C^0$$

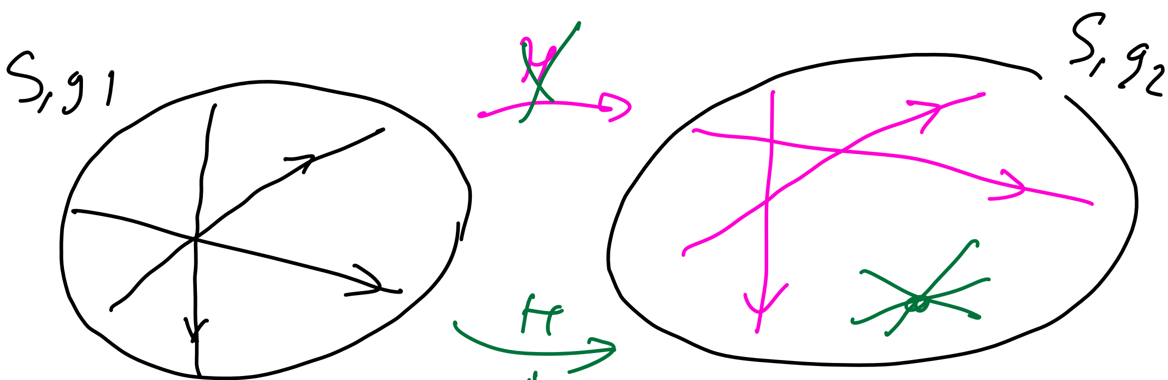
STEP 2: $H^* vol_2 = vol_1$ (Otal)

$$\Updownarrow$$

H is C^∞

$H : T^1S \to T^1S$

STEP 3: H sends ~~send~~ triple intersections to triple intersections.



$\Downarrow$

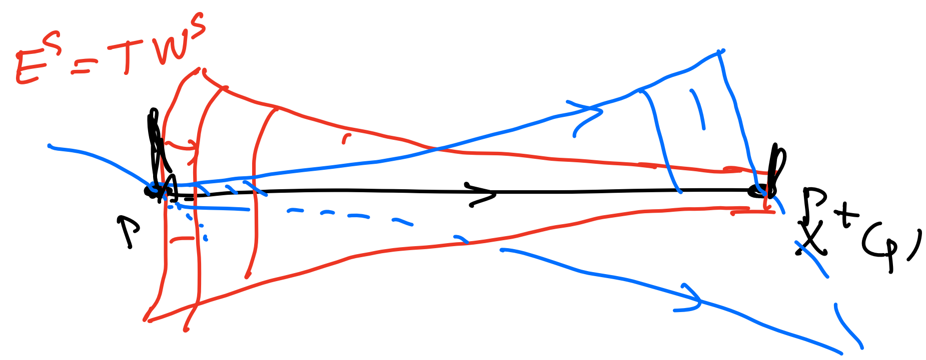
H induces isometry $S_{,g_1} \to S_{,g_2}$.

Anosov flow

Def: $X^t : M \to M$ is Anosov $\exists$ DX^t-invariant splitting $\quad v = \partial X^t$

$\forall p \quad T_pM = E^s_p \oplus X_p \oplus E^u_p$ such that

$$\|DX^t v\| \le C\lambda^t \|v\|, \quad t>0 \qquad v \in E^s$$
$$\|DX^{-t} v\| \le C\lambda^t \|v\|, \quad t>0 \qquad v \in E^u$$



<u>Anosov:</u> If X_1^t, X_2^t $d_{C^1}(X_1, X_2)$ is small then the flows are orbit equivalent.

$\exists$ $h: M \to M$ sends orbits of X_1^t to orbits of X_2^t

$$h(\{X_1^t(p) : t \in \mathbb{R}\}) = \{X_2^t(h(p)) : t \in \mathbb{R}\}$$

$\Uparrow$

<u>Conjugacy:</u> $\underline{H \circ X_1^t = X_2^t \circ H}$

is orbit eq. but not the other way.

<u>Question:</u> X_1^t, X_2^t orbit eq. Are They conjugate C^0, C^1, C^∞?

Obstruction 1: If H is a conj

$$p = X_1^T(p) \Rightarrow H(p) = X_2^T(H(p))$$

periods at all periodic points p.

Obstruction 2: If H is a conj. at H is C^1

$$H \circ X_1^T(p) = X_2^T(H(p))$$

$$DX_1^T(p) = (DH)^{-1} DX_2^T (DH)$$

Hence $DX_1^T : T_pM \circlearrowleft$ is conjugate

DX_2^T hence has the same eigenvalues

dim 3 $\quad DX_1^T = \begin{pmatrix} 1 & & \\ & \lambda_p & \\ & & \mu_p \end{pmatrix}$

$$\lambda_p = \lambda_{H(p)} \quad \text{and} \quad \mu_p = \mu_{H(p)}$$

Countable family of invariants of C^1 conjugacy.

Theorem (89, de la Llave, Marco, Moriyón)

dim $M = 3$ If X_1^t and X_2^t have matching periods and matching eigenvalues

then they are C^∞ conjugate.

Theorem (AG – Federico Rodriguez Hertz)

dim = 3 X_1^t, X_2^t (volume preserving)
and periods are matching ($\Longleftrightarrow$ C^0 conjugacy)
Then either

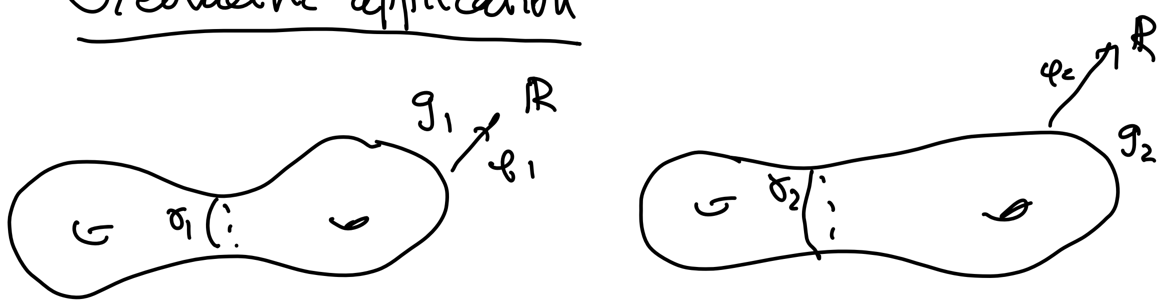
- X_1^t and X_2^t are C^∞ conjugacy.

or

- X_1^t and X_2^t are constant roof suspension Anosov flows

(Feldman–Ornstein: POs ↑ true X_1^t, X_2^t if contact)

Geometric application



Lafont – Khalil: replace (*) by

$$(**) \quad \int_{\gamma_1} \varphi_1(\gamma_1(t))\,dt = \int_{\gamma_2} \varphi_2(\gamma_2(t))\,dt$$

Thm: If (**) $\varphi_1, \varphi_2 \in C^3$ then

$\exists f : (S, g_1) \to (S, g_2)$ isometry. and

$$\varphi_1 = \varphi_2 \circ f.$$

HW: Find two counterexamples.