Cut locus and its applications

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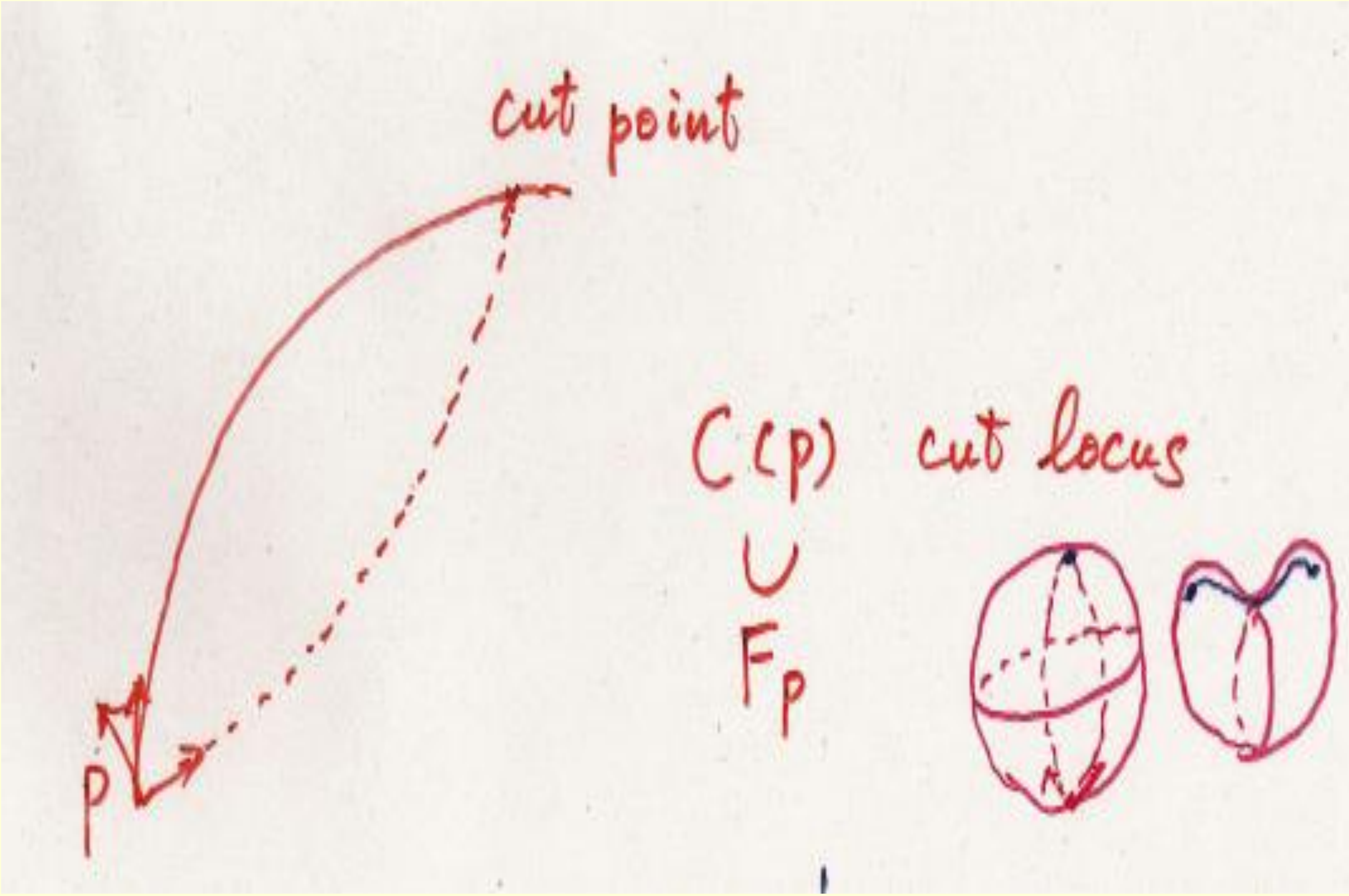

cut point
p
C(p) cut locus
∪
F_p

History of cut locus (1)

- H. Poincare (1905)
- S. Myers (1935-36)

  g: analytic,

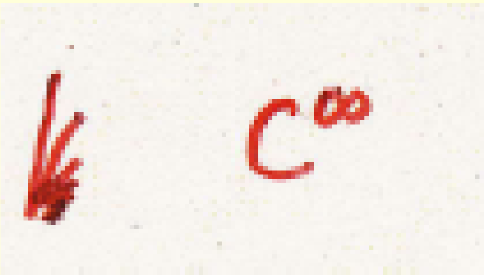
  homeo. to sphere → finite tree

  genus k → 2k cycles

- T. Sakai (1977-78) symmetric space
- H. Gluck & D. Singer (1979)

  g: smooth metric

  with non-triangulable cut locus



- K. Shiohama & M. Tanaka (1996)

  Alexandrov surface

distance function to the cut point on S_p(M)

- J. Hebda (1994), J. I___ (1996)
  absolutely continuous
  Ambrose’s problem (surface)

- J. I__ & M. Tanaka (2001)
  Lipschitz continuous
  (L. Nirenberg & Y. Li)

- M. Berger (2000)

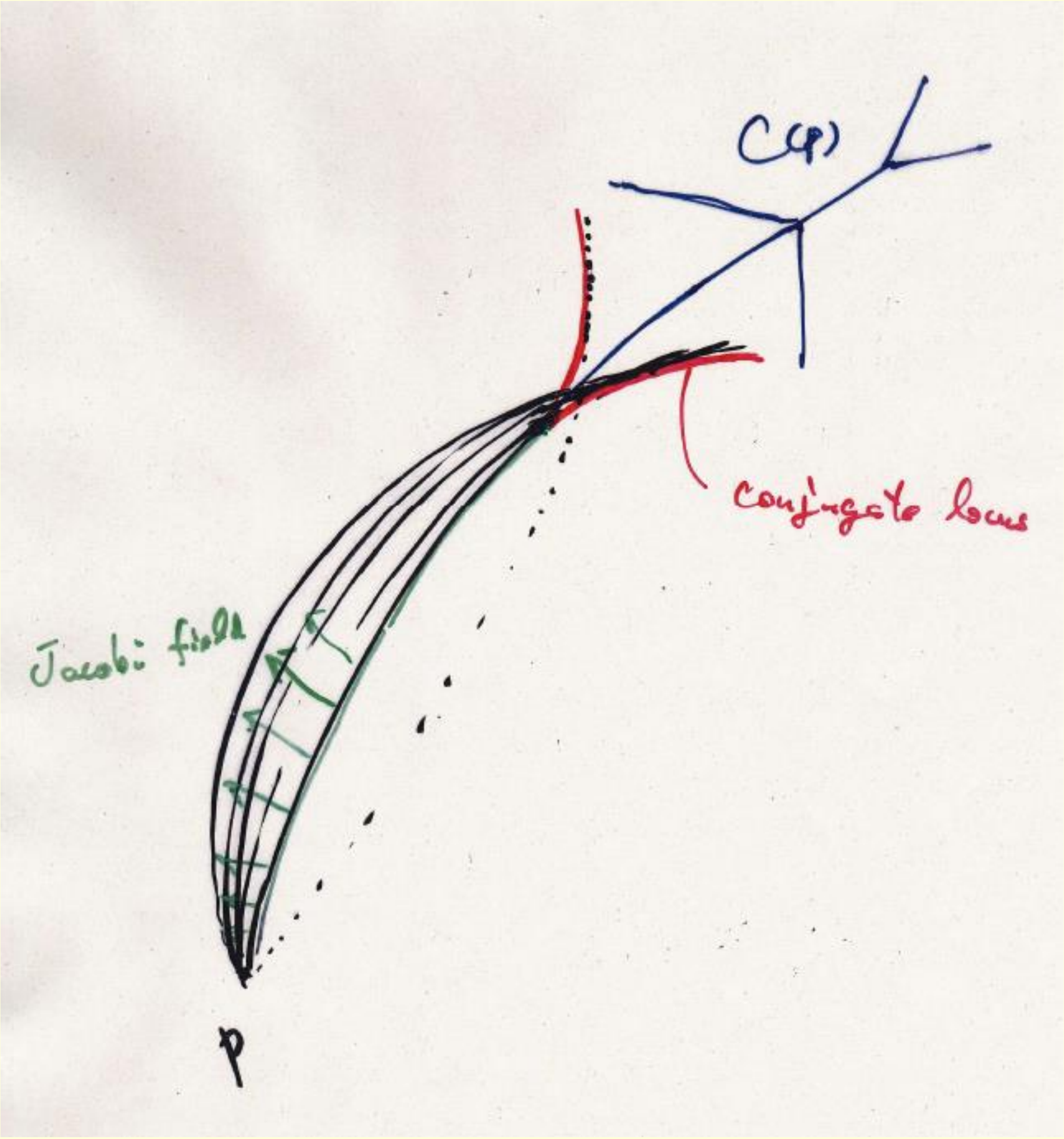
  “Riem. geom. during the 2’nd half of 20 C.”

  Jacob’s last statement is unproved

- R. Sinclair & M. Tanaka “Loki”
- J. I___ & R. Sinclair “Thaw”

- J. I___ & K. Kiyohara (2004)

  Ellipsoid case


C(p)
conjugate locus
Jacobi field
p

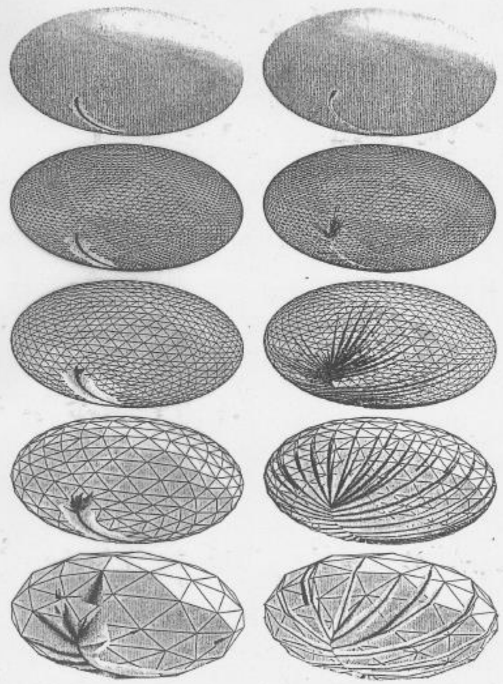


Figure 6: Approximations to the cut locus of an ellipsoid from a non-symmetrically placed starting point. $\Delta\theta = 0.5$ and 0.05 in the left and right columns respectively. The number of triangles increases as one moves up each column, from 128 to 32768.

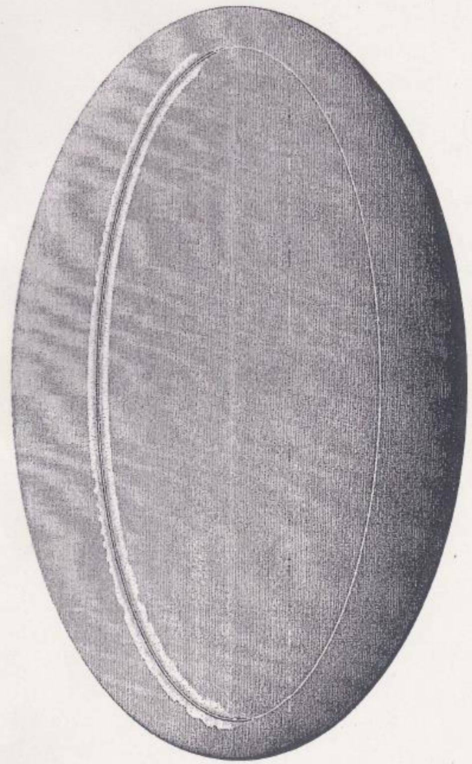


Figure 15: The cut locus from the point with elliptic coordinates $(u_1, u_2) = (0.2371636915, 0.9296614183)$ on the surface given by $x_0^2/0.04 + x_1^2/0.36 + x_2^2 = 1$ as computed by Thaw, beneath the white curvature line given by $u_1 = 0.2371635554$.

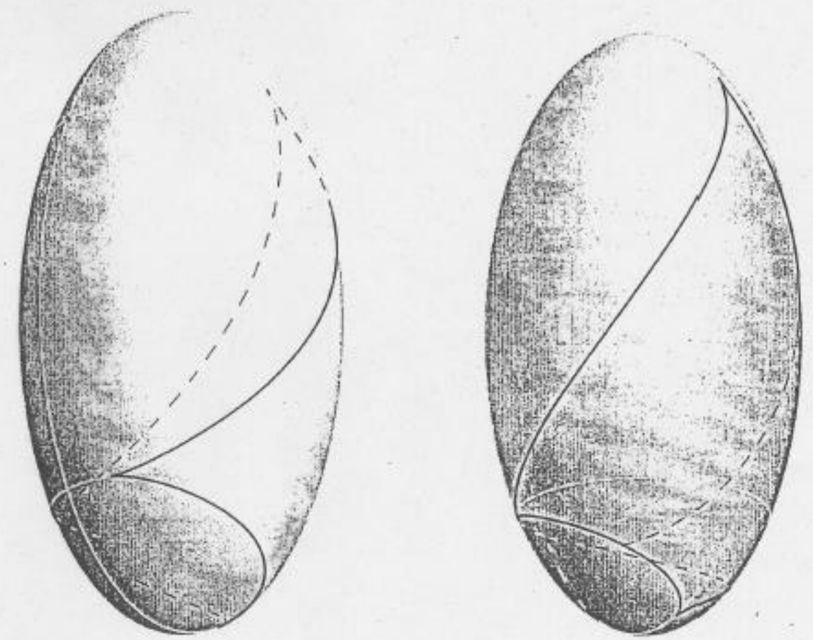
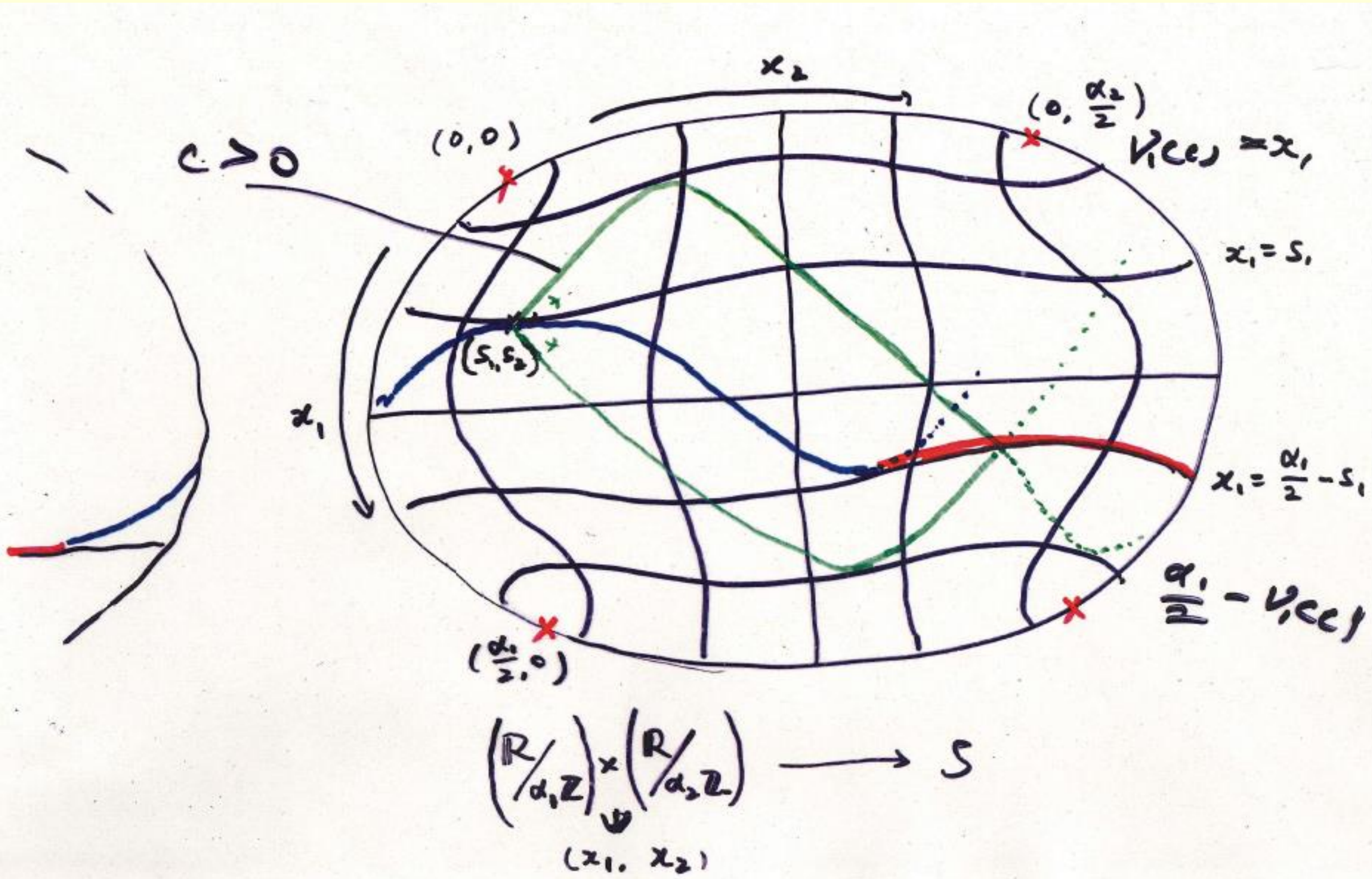


Figure 5: Two views of the conjugate locus (first caustic) from a point on a prolate ellipsoid. The starting point is shown as a black cross where it is in the foreground, and white where it is on the far side of the surface.



x_2
$(0, \frac{\alpha_2}{2})$
$(0,0)$
$c > 0$
$V_1(c) = x_1$
$x_1 = s_1$
(s_1, s_2)
α_1
$x_1 = \frac{\alpha_1}{2} - s_1$
$\frac{\alpha_1}{2} - V_1(c)$
$(\frac{\alpha_1}{2}, 0)$
$(\mathbb{R}/\alpha_1\mathbb{Z}) \times (\mathbb{R}/\alpha_2\mathbb{Z}) \longrightarrow S$
(x_1, x_2)

Ellipsoids in general dimension

Joint work with K. Kiyohara

[1] Cut loci and conjugate loci on Liouville surfaces. *Manuscripta Math.* 136 (2011), no. 1-2, 115–141.

[2] The cut loci on ellipsoids and certain Liouville manifolds. *Asian J. Math.* 14 (2010), no. 2, 257–289.

[3] Cut loci and conjugate loci on Liouville surfaces, preprint.

M : ellipsoid

$$\sum_{i=0}^{n} \frac{{u_i}^2}{a_i} = 1 \qquad (0 < a_n < \cdots < a_0)$$

$$\mathrm{J} := \left\{ (u_0, \cdots, u_n) \in \mathrm{M} \,\middle|\, u_{n-1} = 0, \sum_{i \neq n-1} \frac{{u_i}^2}{a_i - a_{i-1}} = 1 \right\}$$

C(p) : cut locus of p

K_1(p) : 1st. conjugate locus

Theorem 1

(1) p $\notin$ J ⇨

C(p) is diffeo. to (n-1) dim. closed disk

C(p) $\ni$ p^* : anti podal point of p

C(p) $\subset$ $\{\text{an elliptic coord.} = \text{const.}\}$

(2) p $\in$ J ⇨

C(p) is diffeo. to (n-2) dim. closed disk

C(p) : cod. 1 submanifold in $\{u_{n-1} = 0\}$

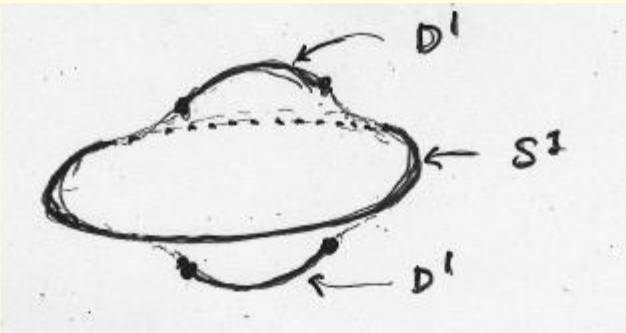
Theorem 2

p ∈ M ($u_i \neq 0$) ⇨

sing (K_1(p)) : 3 conn. Components

one cuspidal edge $\cong n - 2$ dim. sphere

two cuspidal edges $\cong n - 2$ dim. disks



Theorem 3

M $\cong S^n$ (close to round sphere) ⇨

Sing(K_1(p)) $(2 \le i \le n-2)$:

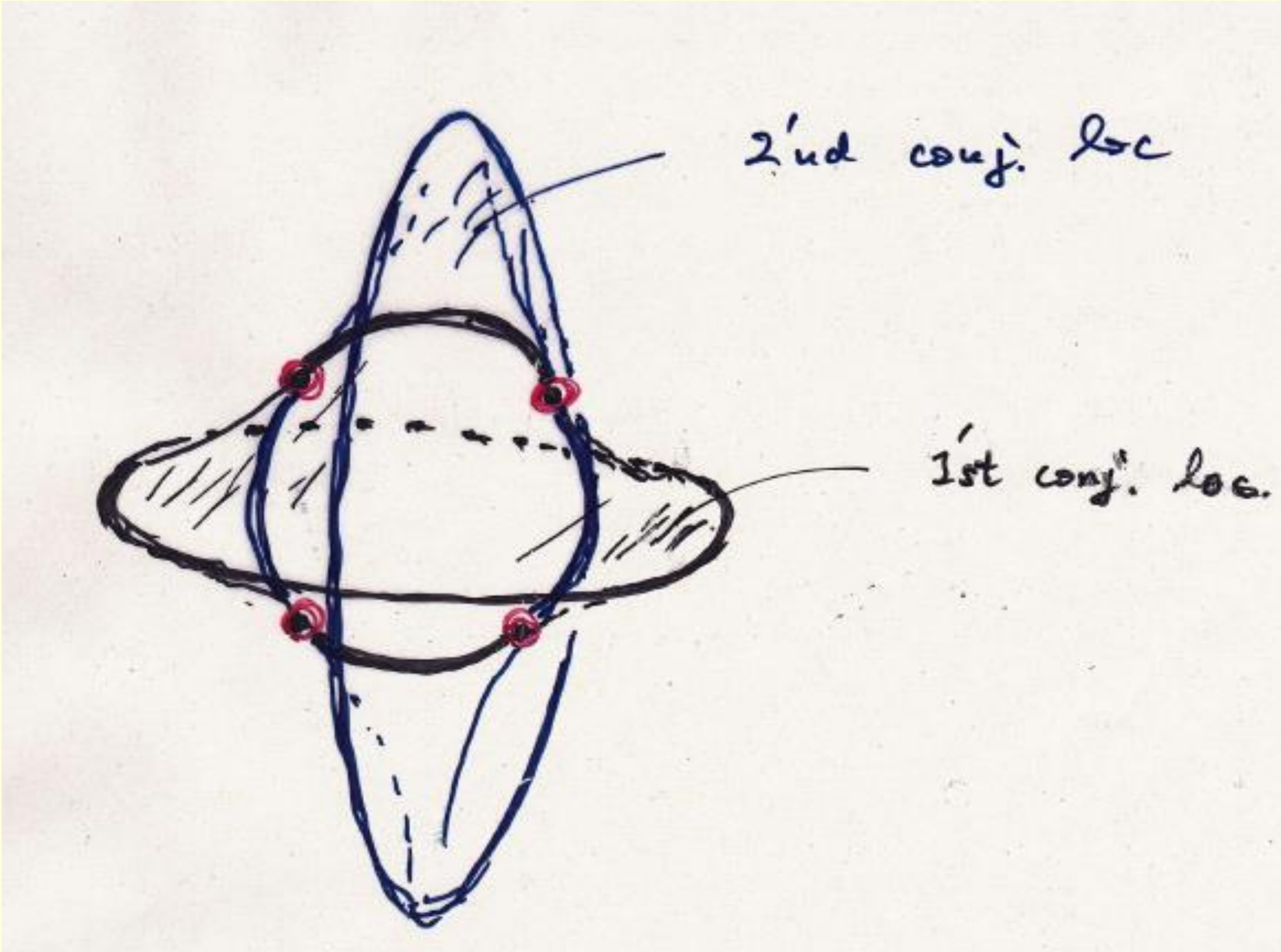
2 conn. components

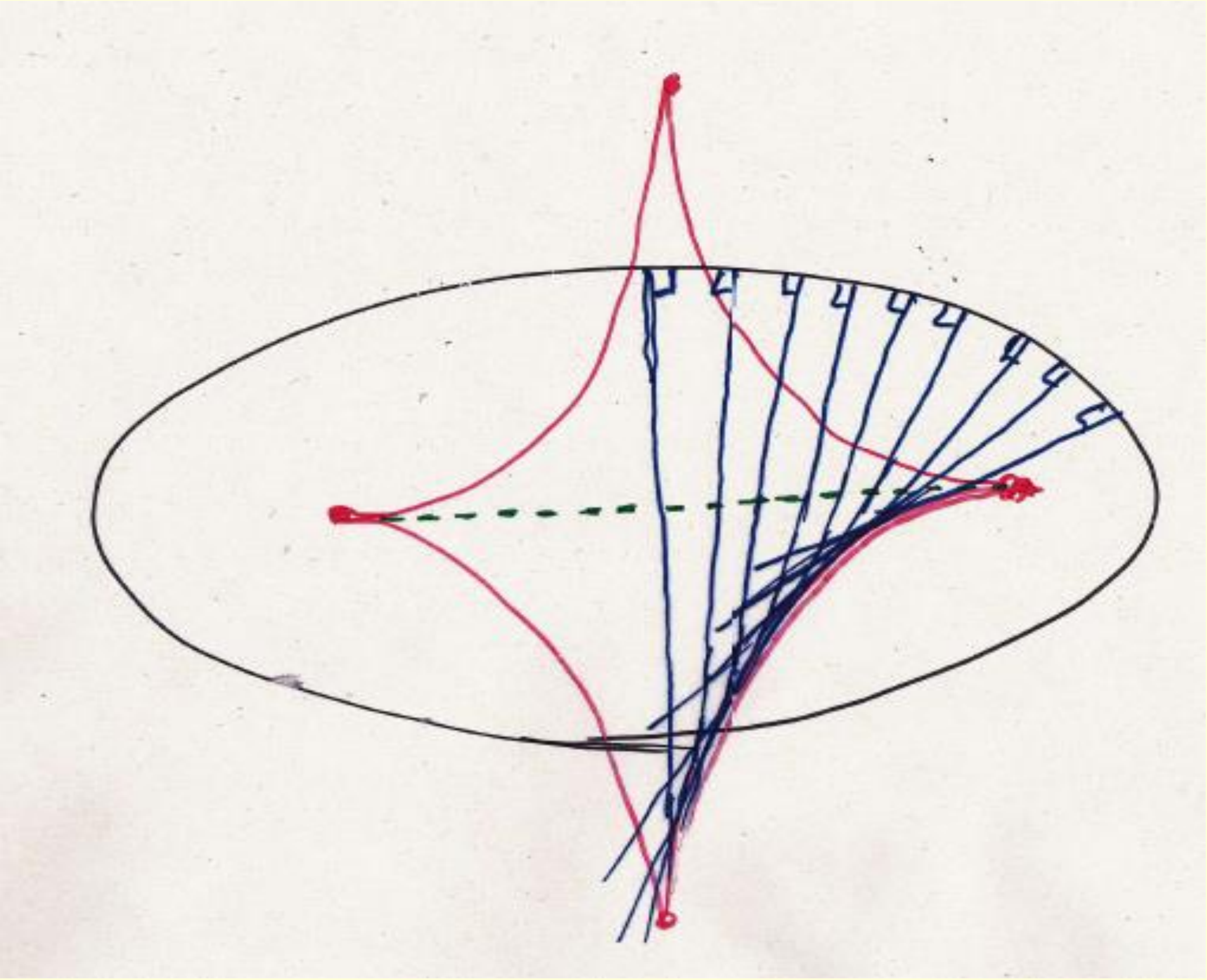
a cuspidal edge $\cong S^{n-1-i} \times D^{i-1}$

$\cong D^{n-1-i} \times S^{i-1}$

K_i(p) ∩ K_{i+1}(p) = Sing(K_i(p)) ∩Sing(K_{i+1}(p))

$\cong S^{n-2-i} \times S^{i-1}$


2'nd conj. loc
1'st conj. loc.



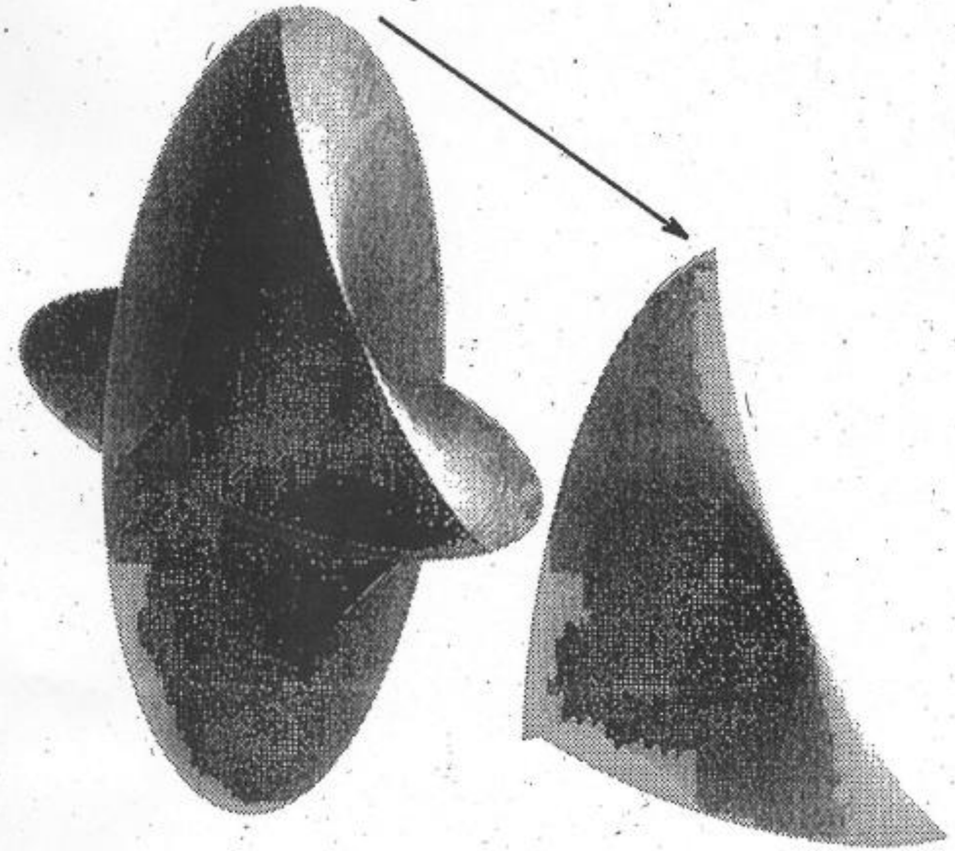


FIGURE 1. Astroïde de Cayley, enveloppe des normales à un ellipsoïde triaxial de l'espace $\mathbb{R}^3$. Un octant a été découpé et translaté pour une meilleure compréhension de la surface.

Fractal cut locus (j. w. with S. Sabau)

Theorem 1

$2 \leq \forall k < \infty$,

$\exists$ Riem. metric at least k-differentiable on $S^{n(k)}$

$\exists$ p $\in$ $S^{n(k)}$ s.t.

1 < Hausdorff dim. of C(p) < 2,

where $n(k) = \dfrac{3^{k+1}}{2} + 1$.

Theorem 2

$2 \leq \forall k < \infty$, under magnetic fields β on $S^{n(k)}$

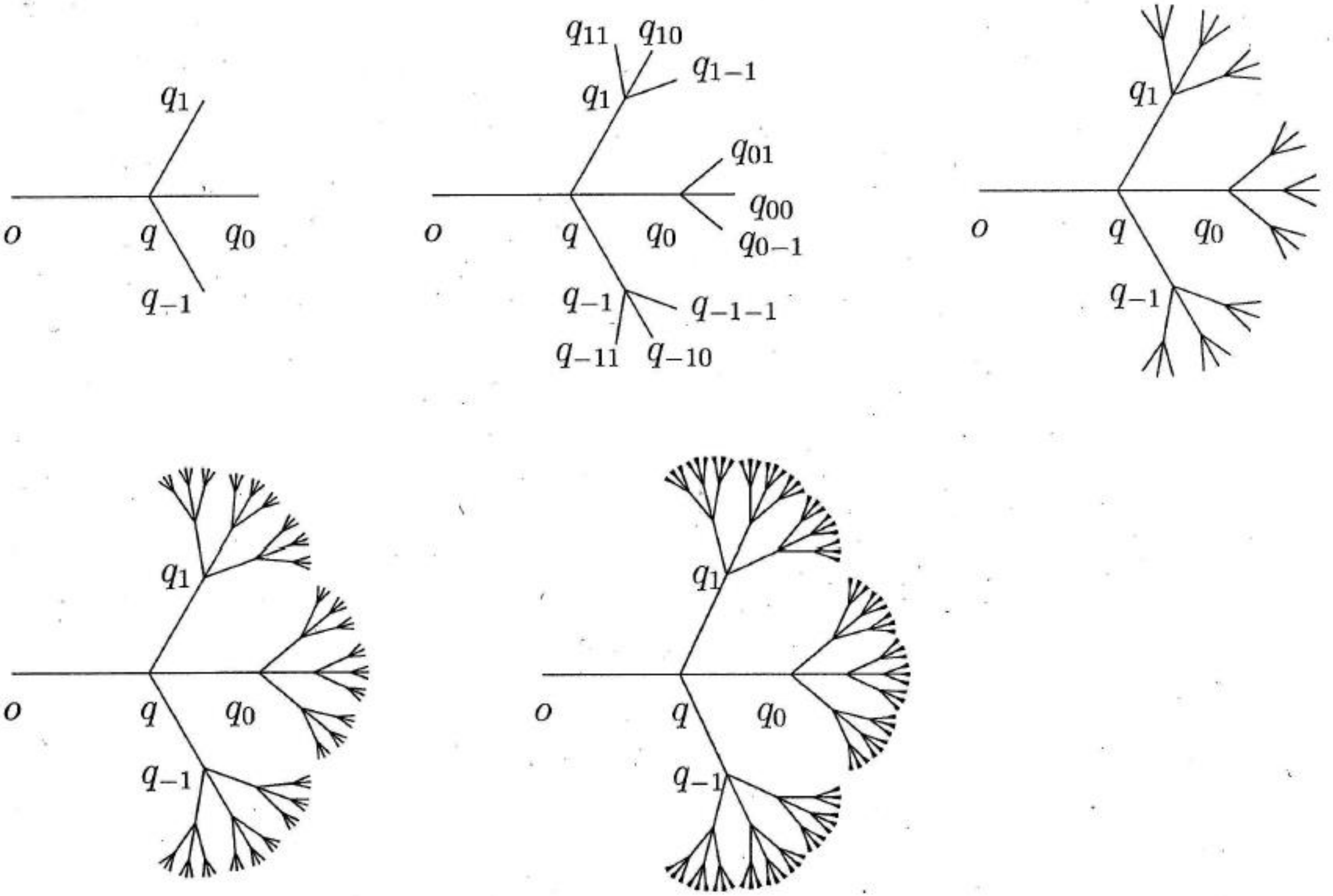
$\exists$ non-Riem., Finsler metric of Randers type

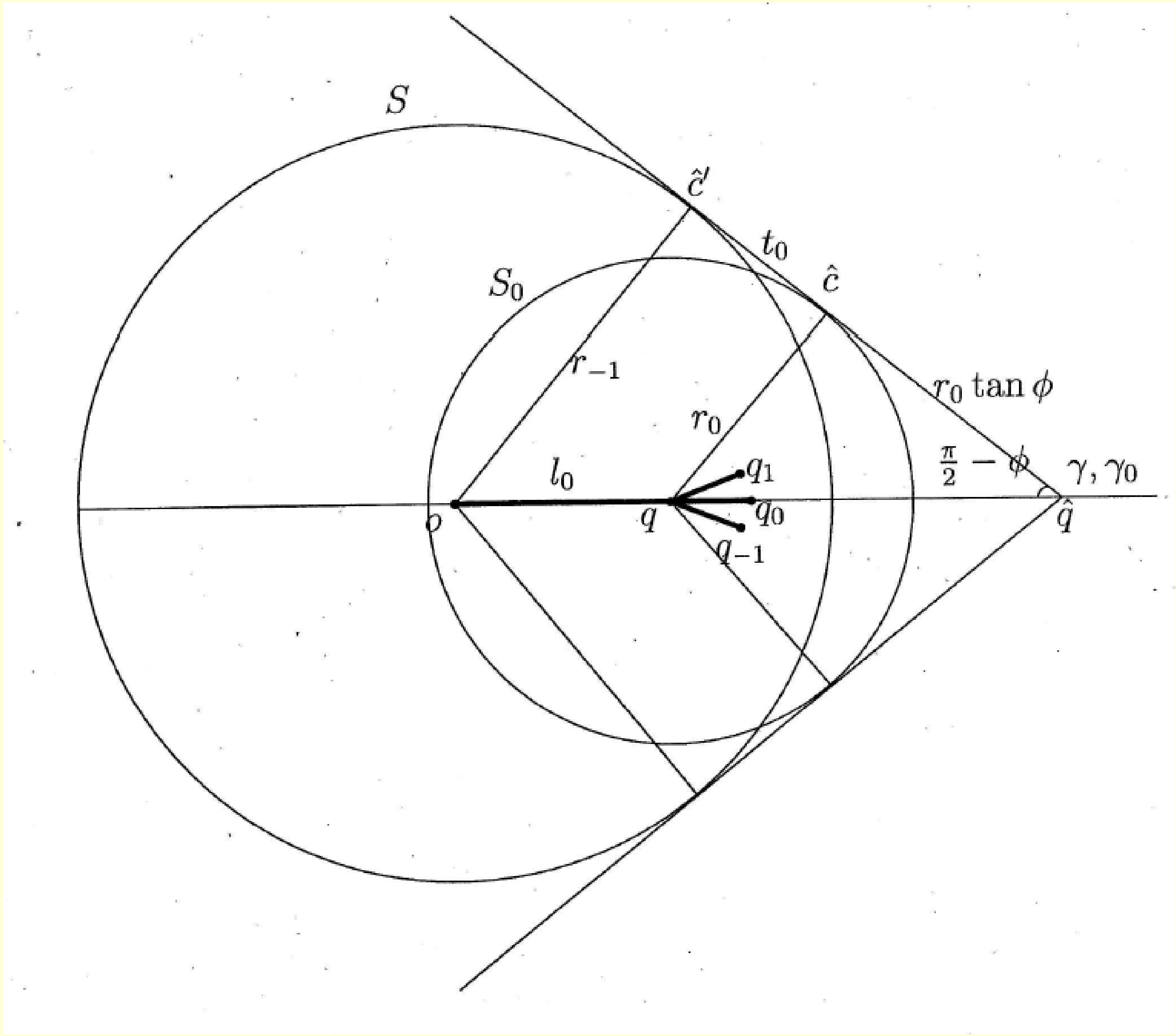
at least k-differentiable on $S^{n(k)}$

$\exists$ p $\in$ $S^{n(k)}$ s.t.

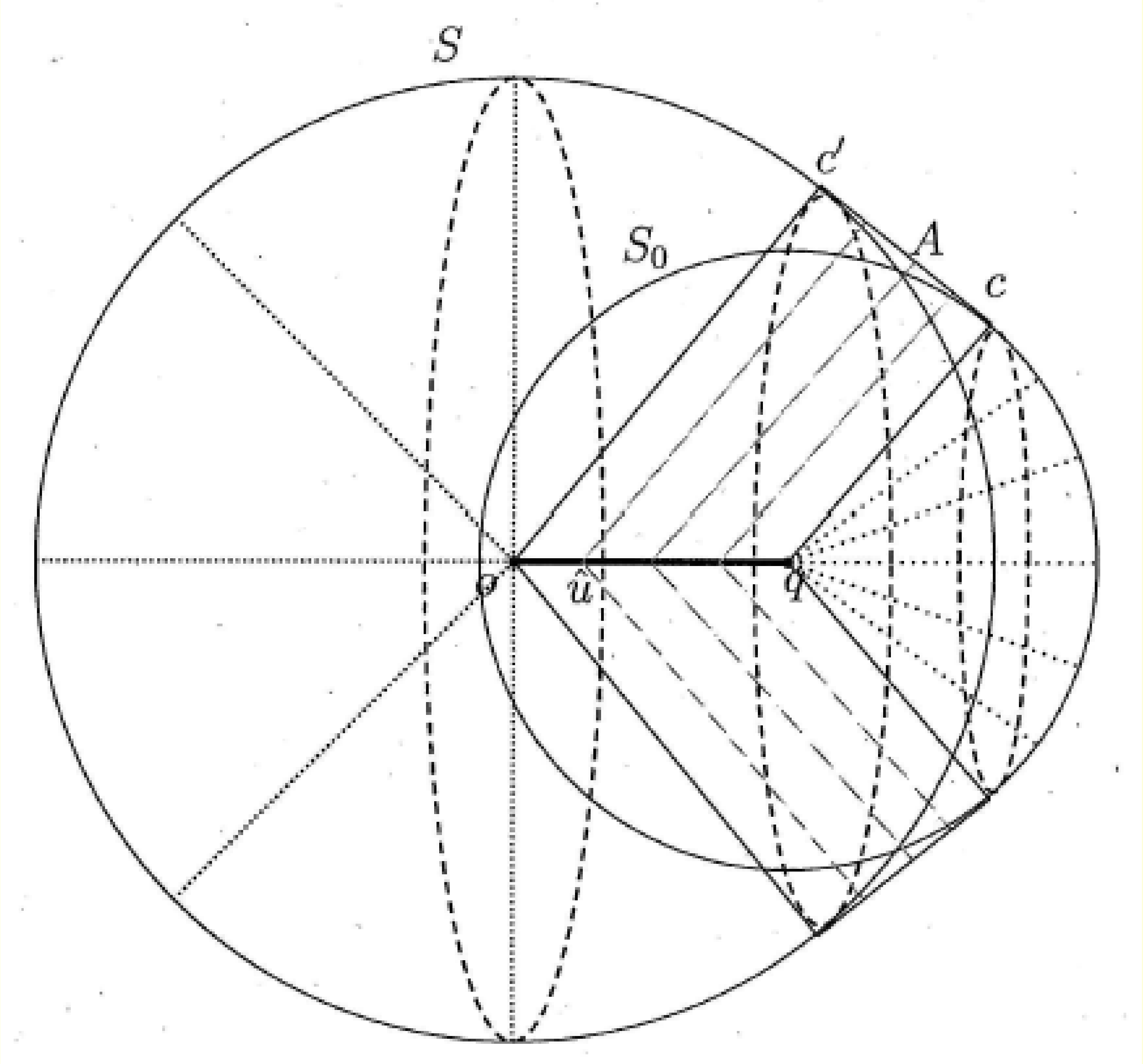
1 < Hausdorff dim. of C(p) < 2,

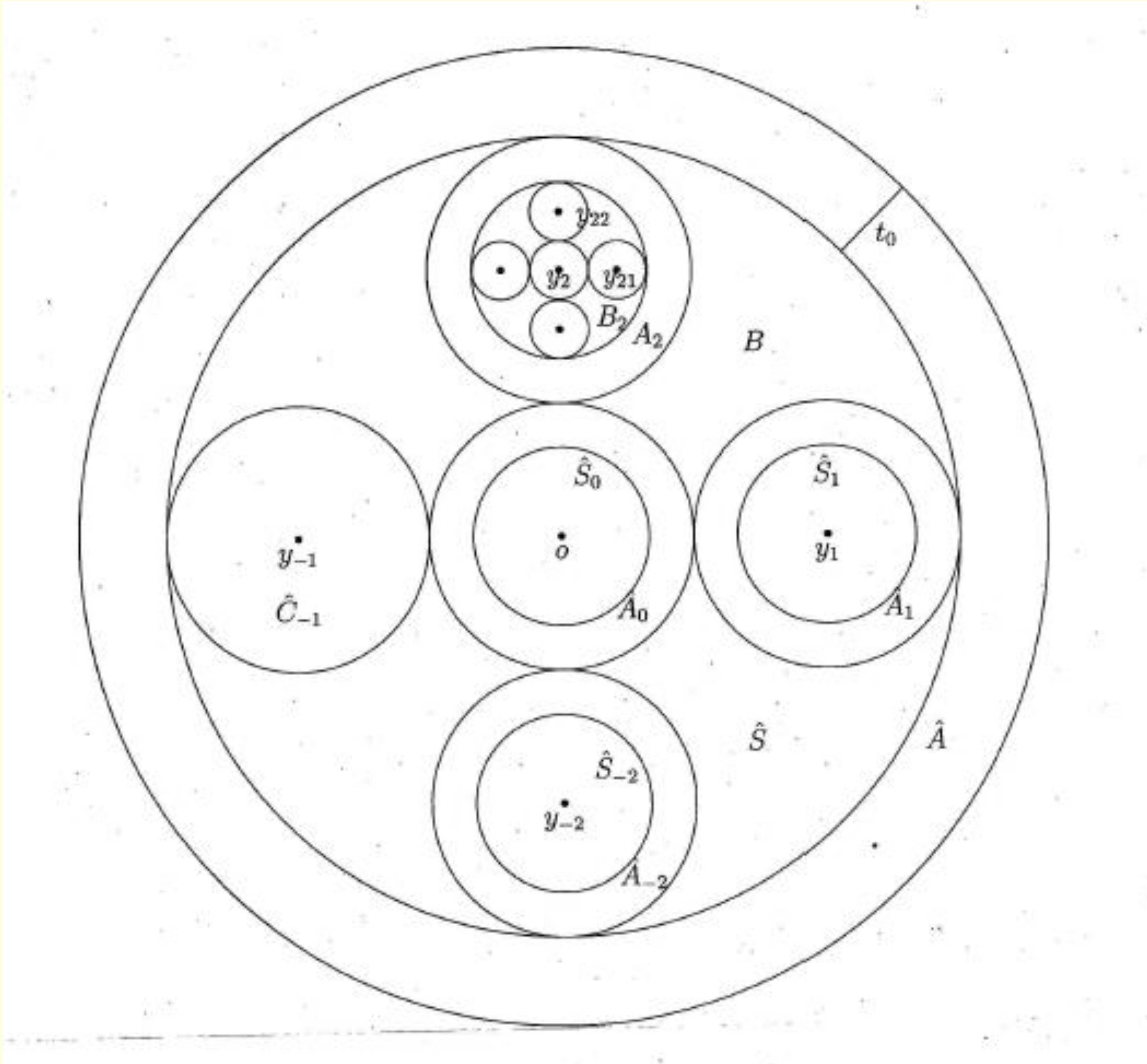
where $n(k) = \frac{3^{k+1}}{2} + 1.$

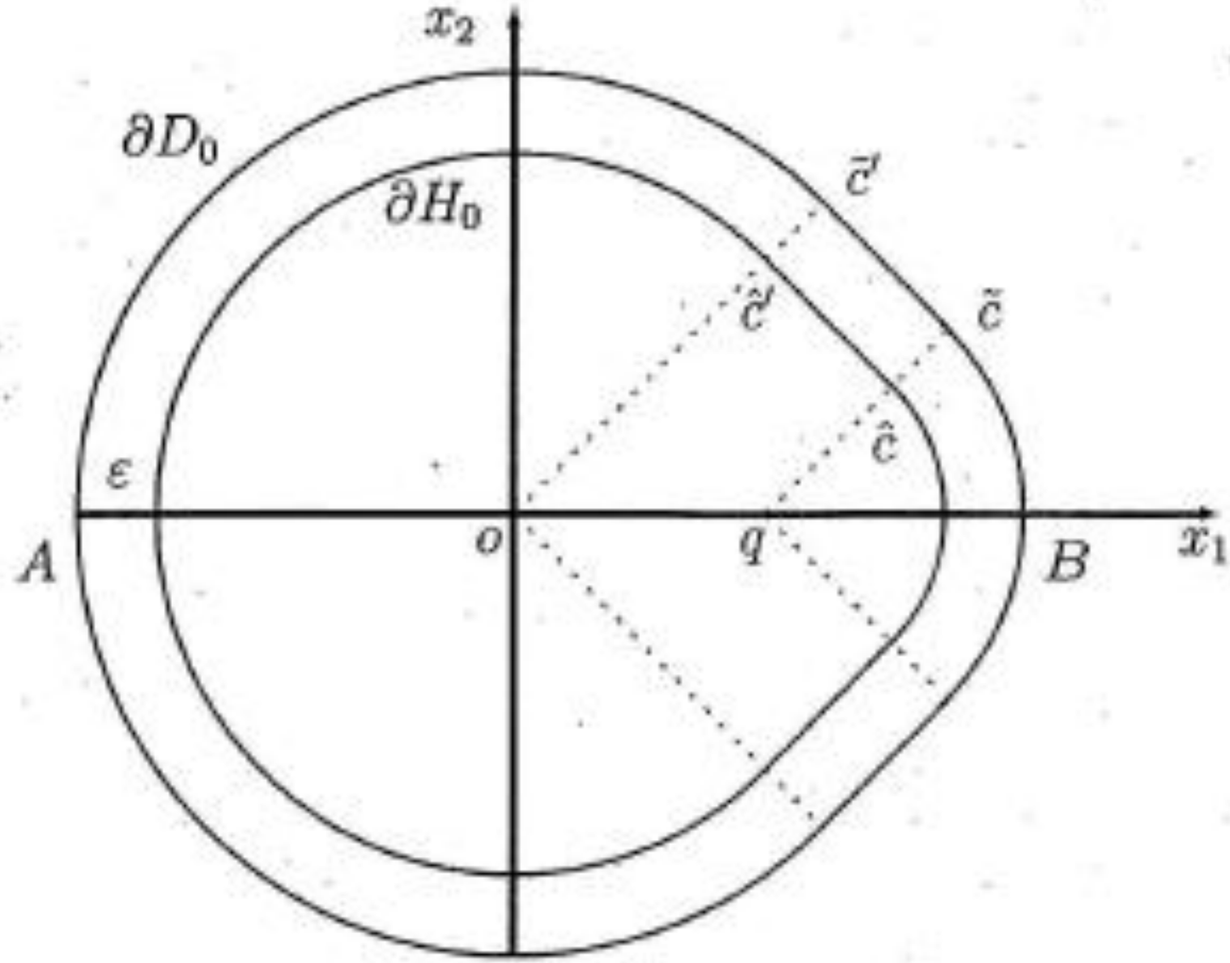

q_1
o
q
q_0
q_{-1}
q_{11}
q_{10}
q_{1-1}
q_1
q_{01}
q_{00}
o
q
q_0
q_{0-1}
q_{-1}
q_{-1-1}
q_{-11}
q_{-10}
q_1
o
q
q_0
q_{-1}
q_1
o
q
q_0
q_{-1}
q_1
o
q
q_0
q_{-1}

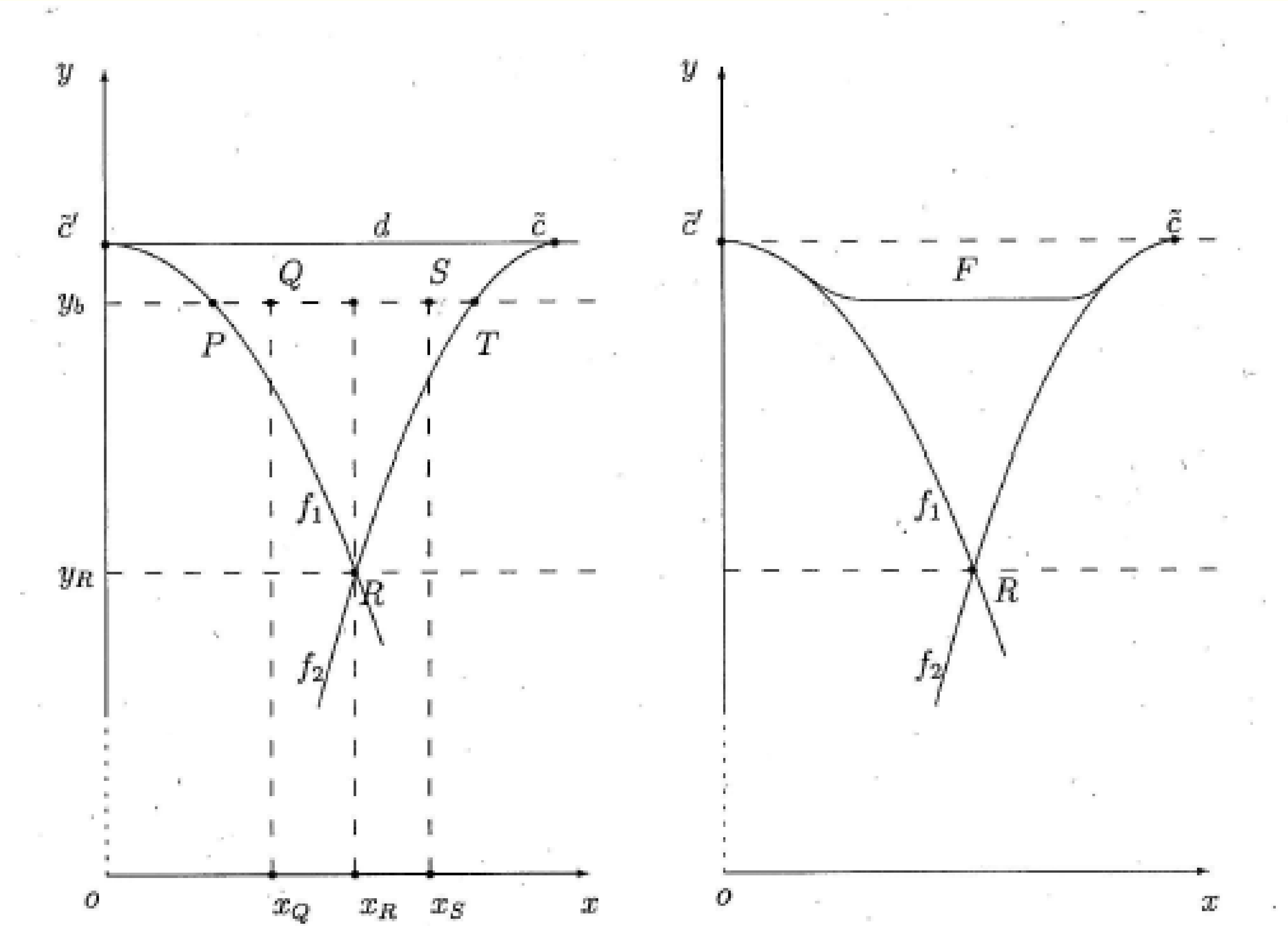


S
$\hat{c}'$
t_0
S_0
$\hat{c}$
r_{-1}
$r_0 \tan\phi$
r_0
q_1
$\frac{\pi}{2} - \phi$
γ, γ_0
l_0
o
q
q_0
$\hat{q}$
q_{-1}

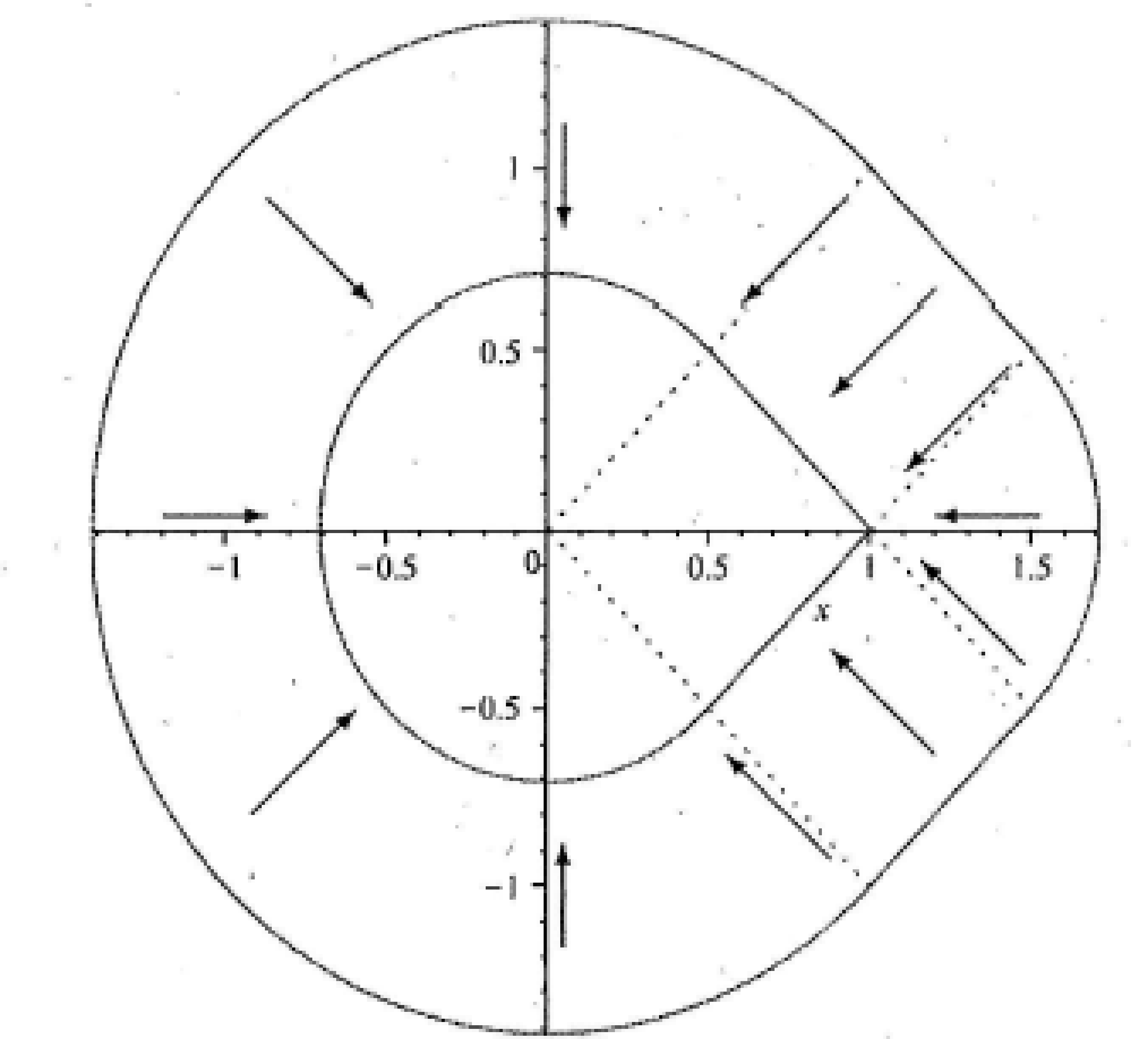

S
c′
S_0
A
c
o
û
q


y_{22}
t_0
y_2
y_{21}
B_2
A_2
B
$\hat{S}_0$
$\hat{S}_1$
y_{-1}
o
y_1
$\hat{C}_{-1}$
A_0
$\hat{A}_1$
$\hat{S}$
$\hat{A}$
$\hat{S}_{-2}$
y_{-2}
$\hat{A}_{-2}$

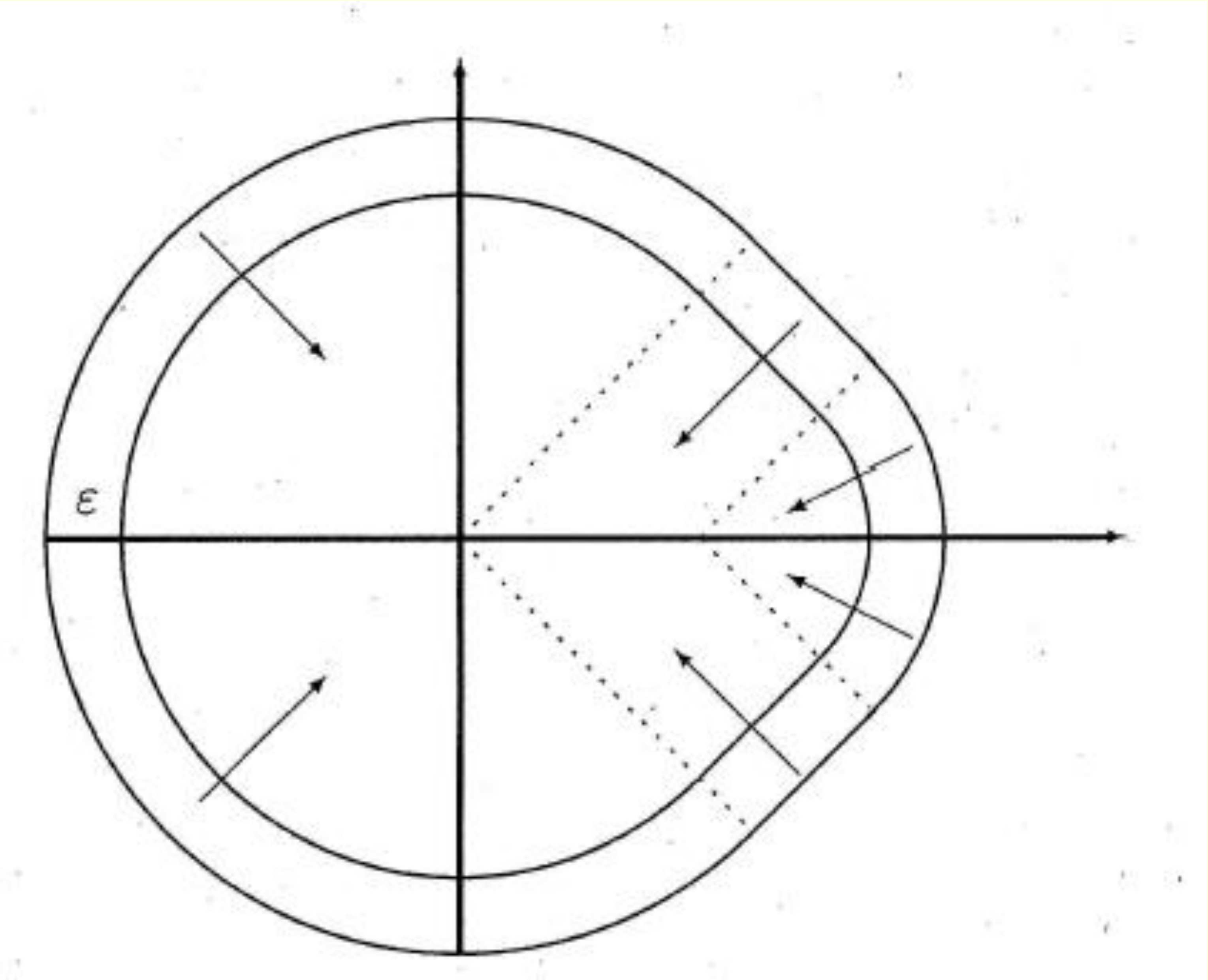

x_2
∂D_0
∂H_0
$\vec{c}'$
$\hat{c}'$
$\tilde{c}$
$\hat{c}$
ε
A
o
q
B
x_1

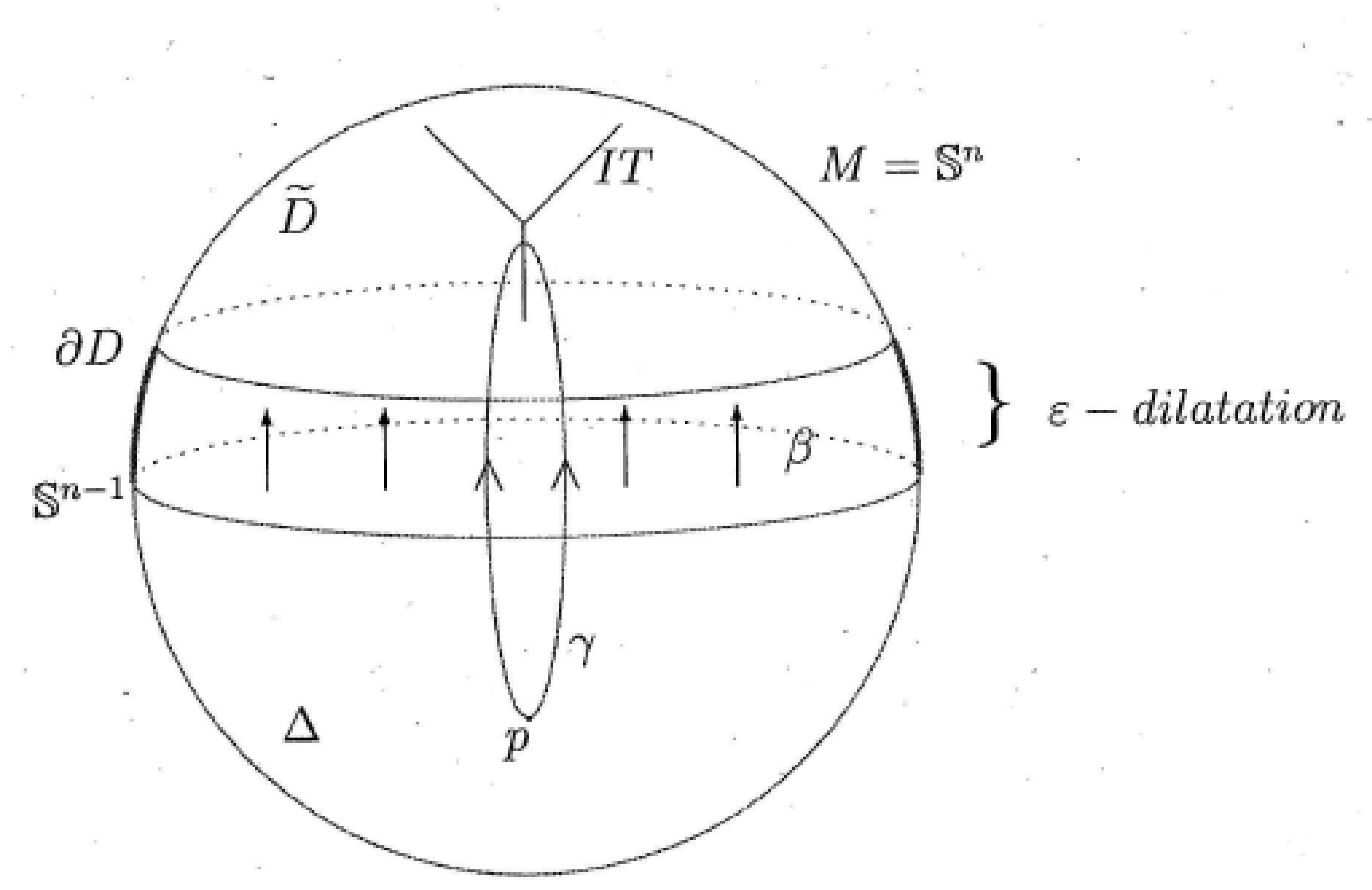


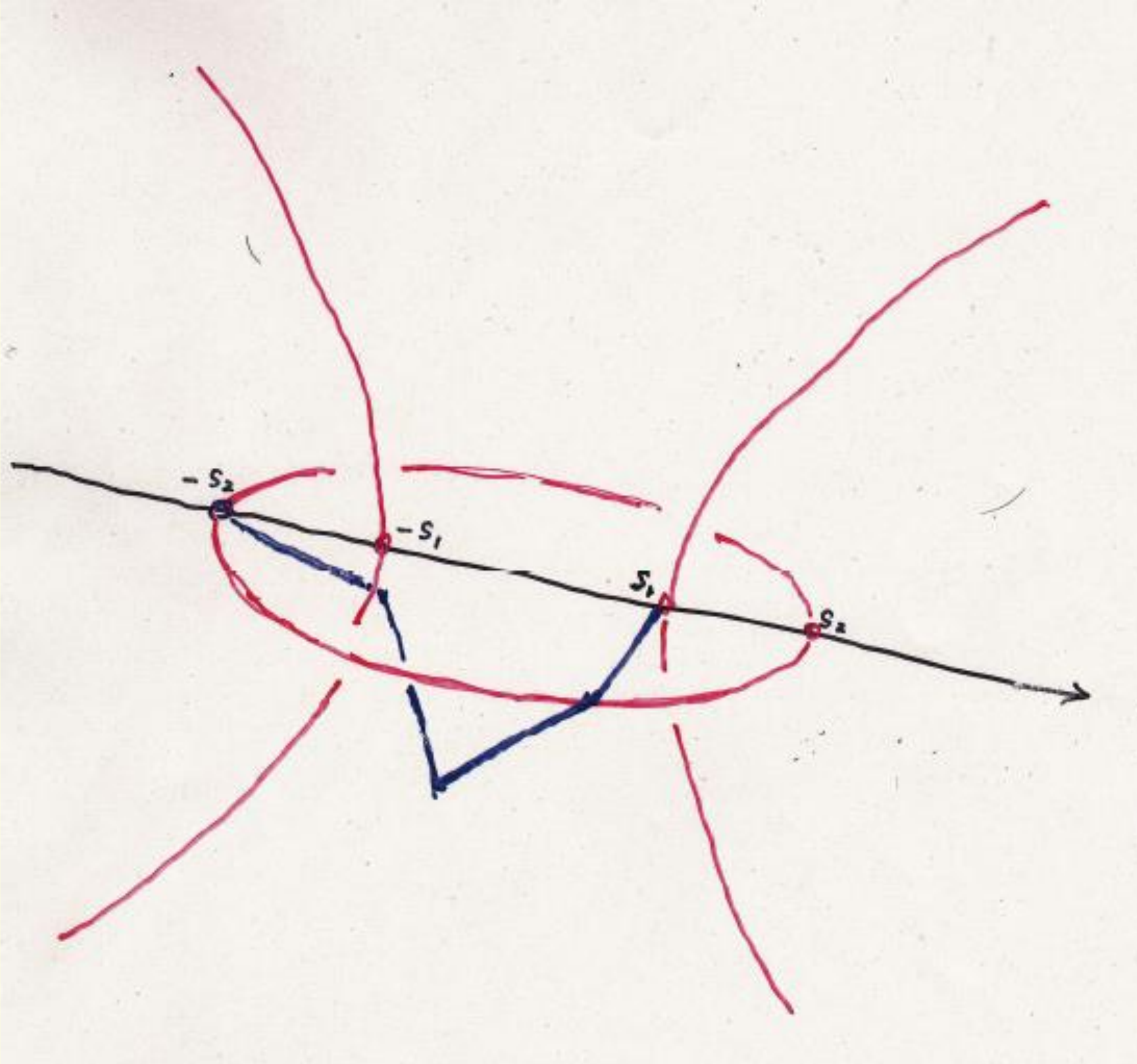
y
$\tilde{c}'$
d
$\tilde{c}$
Q
S
y_b
P
T
f_1
y_R
R
f_2
o
x_Q
x_R
x_S
x
y
$\tilde{c}'$
$\tilde{c}$
F
f_1
R
f_2
o
x

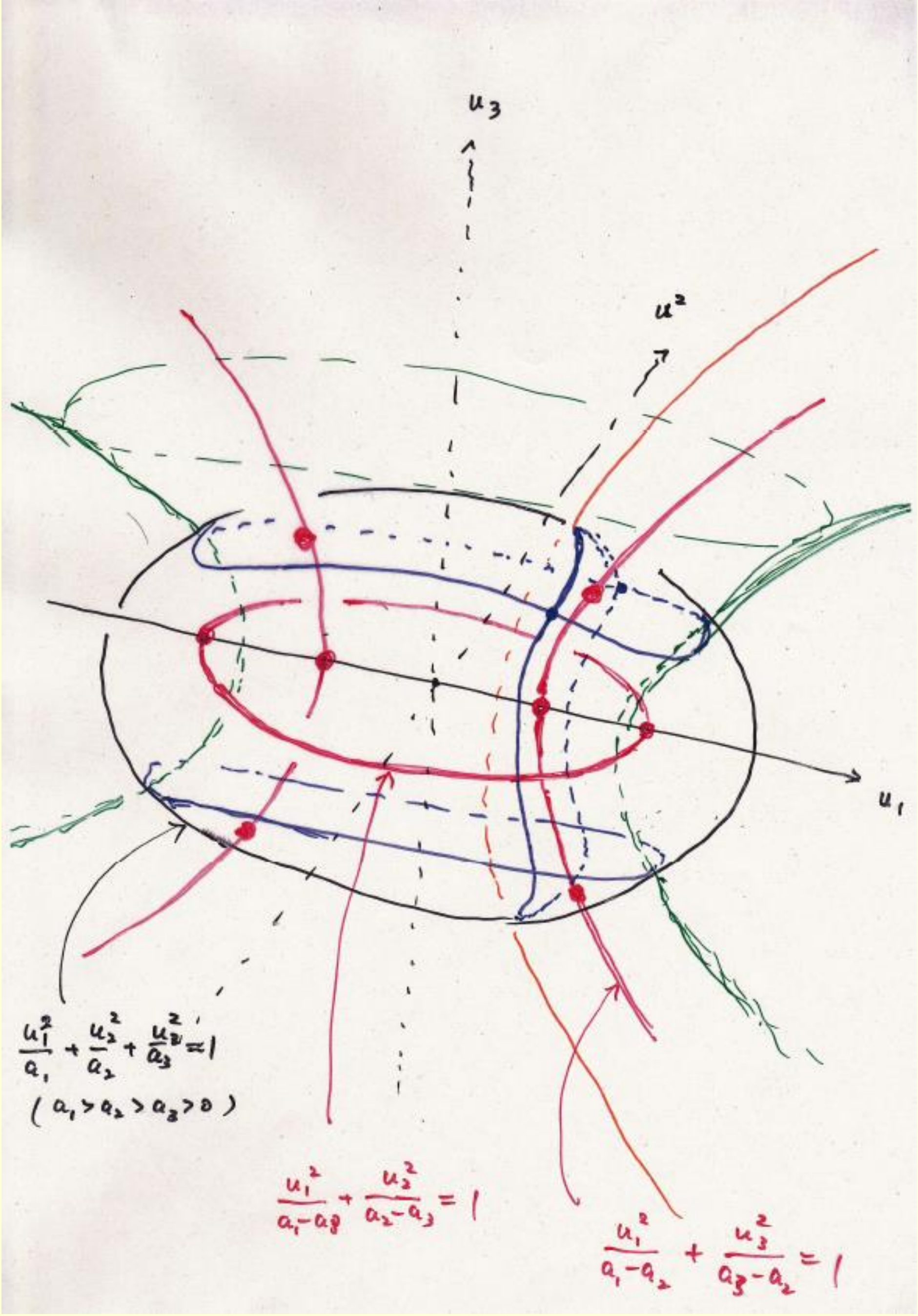


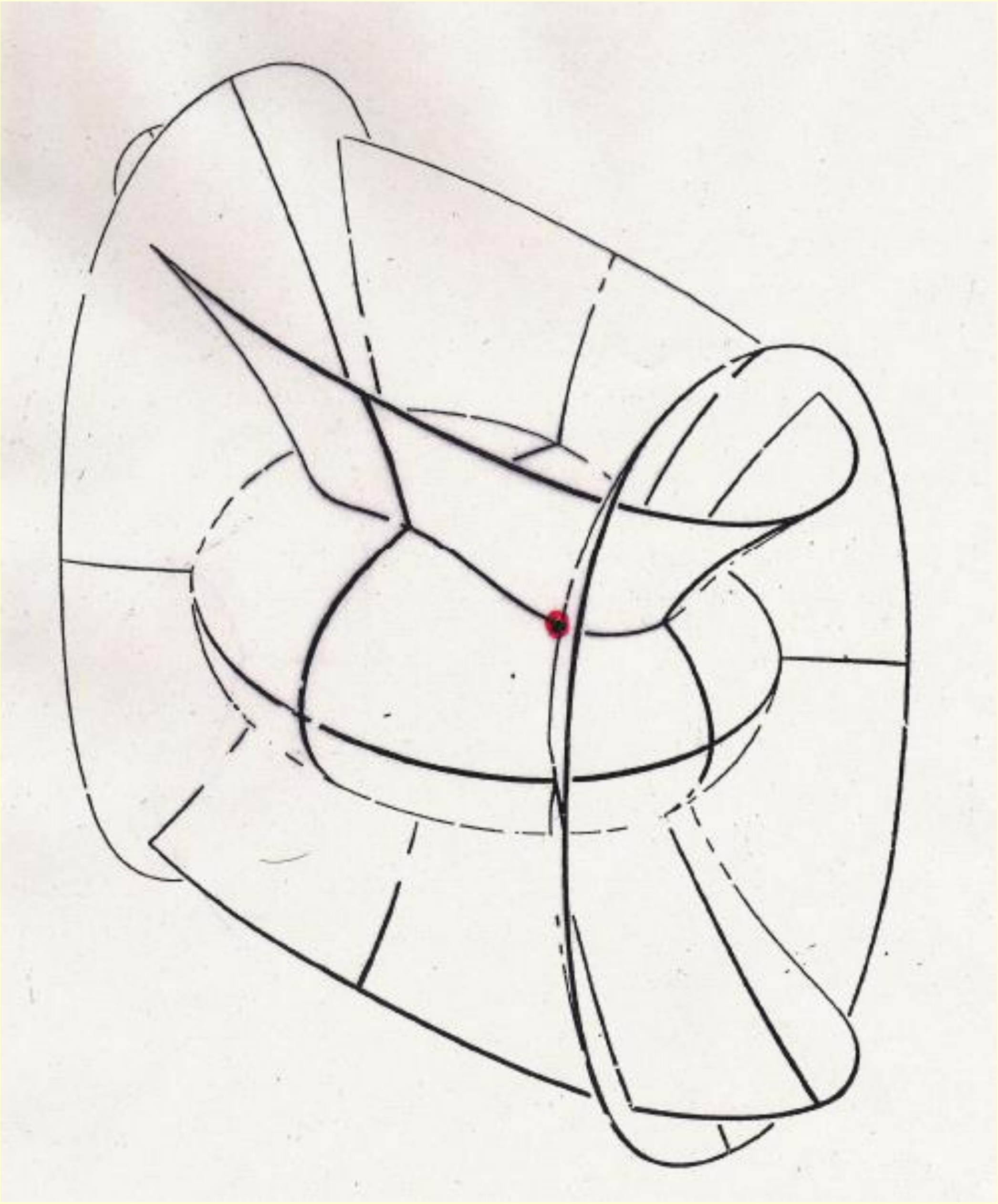
1
0.5
−0.5
−1
−1
−0.5
0
0.5
1
1.5
x

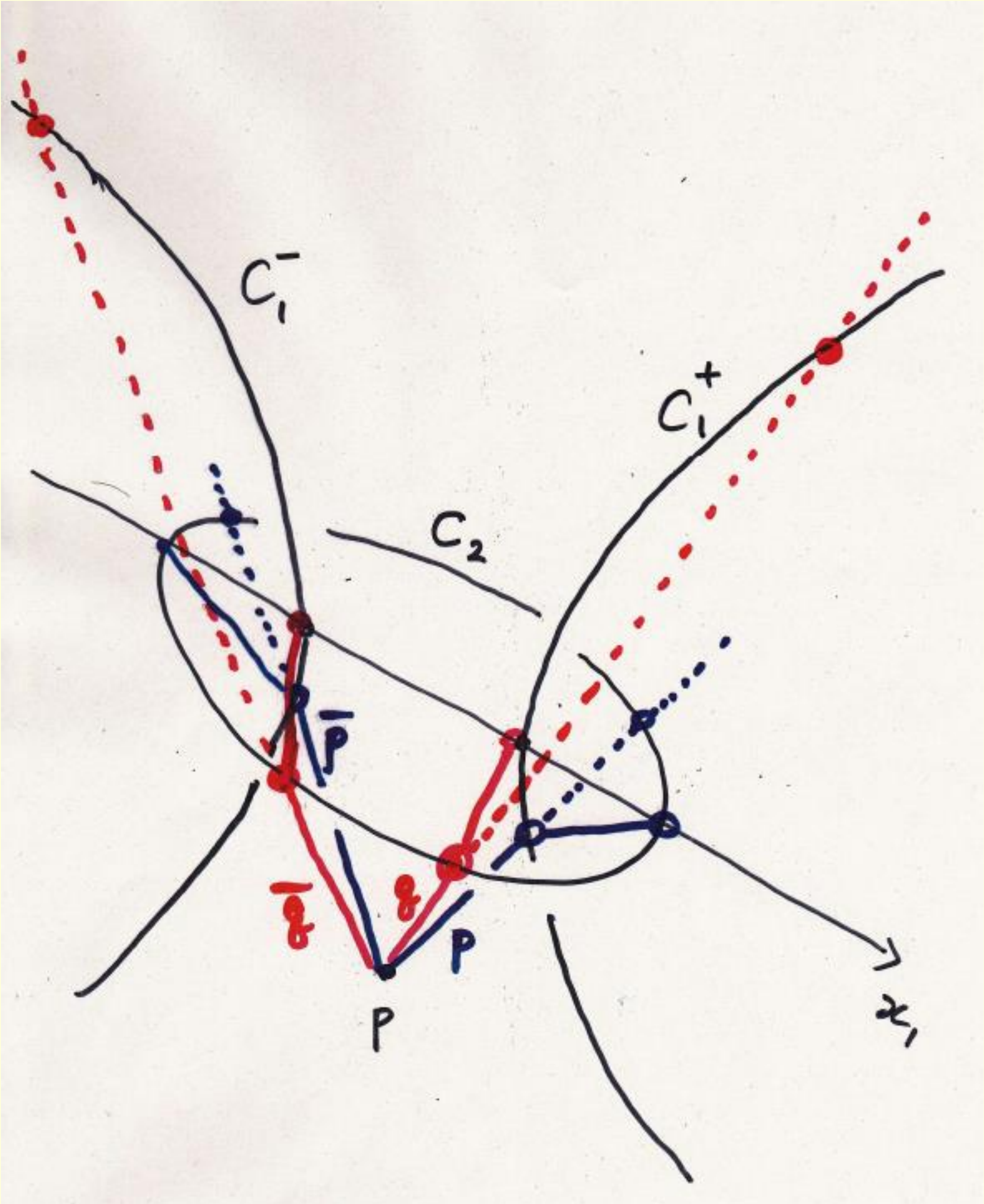

ε


IT
$M = \mathbb{S}^n$
$\widetilde{D}$
∂D
$\varepsilon - dilatation$
β
$\mathbb{S}^{n-1}$
γ
Δ
p

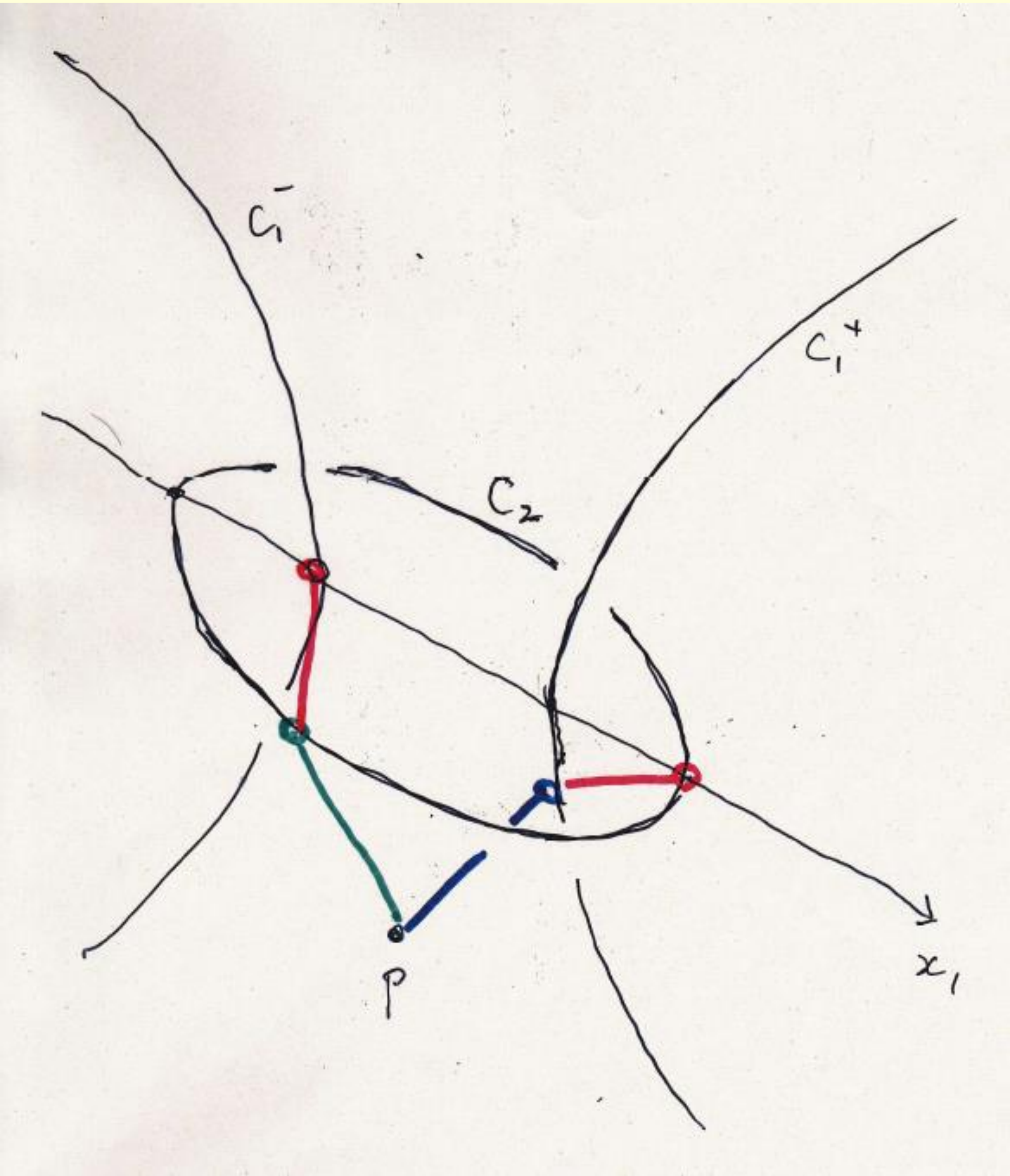

-S2
-S1
S1
S2

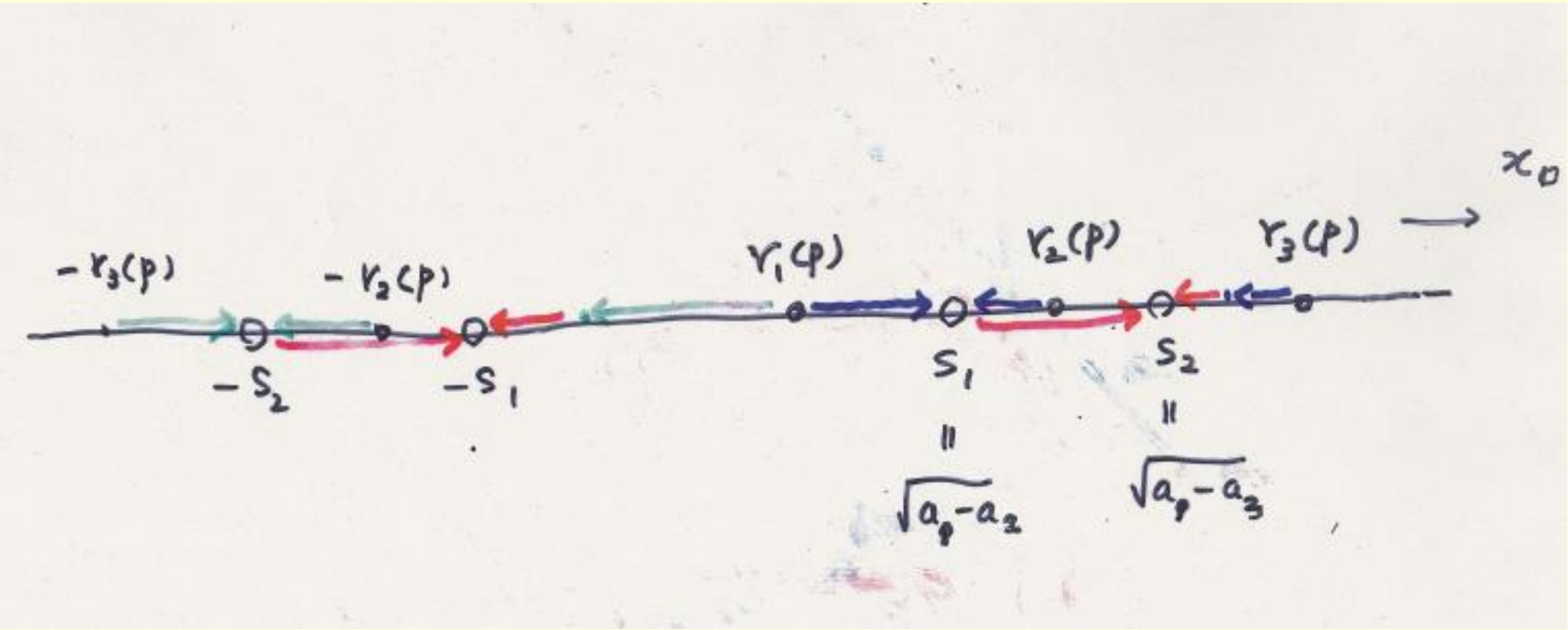

u_3
u^2
u_1
$\frac{u_1^2}{a_1}+\frac{u_2^2}{a_2}+\frac{u_3^2}{a_3}=1$
$(a_1>a_2>a_3>0)$
$\frac{u_1^2}{a_1-a_3}+\frac{u_2^2}{a_2-a_3}=1$
$\frac{u_1^2}{a_1-a_2}+\frac{u_3^2}{a_3-a_2}=1$



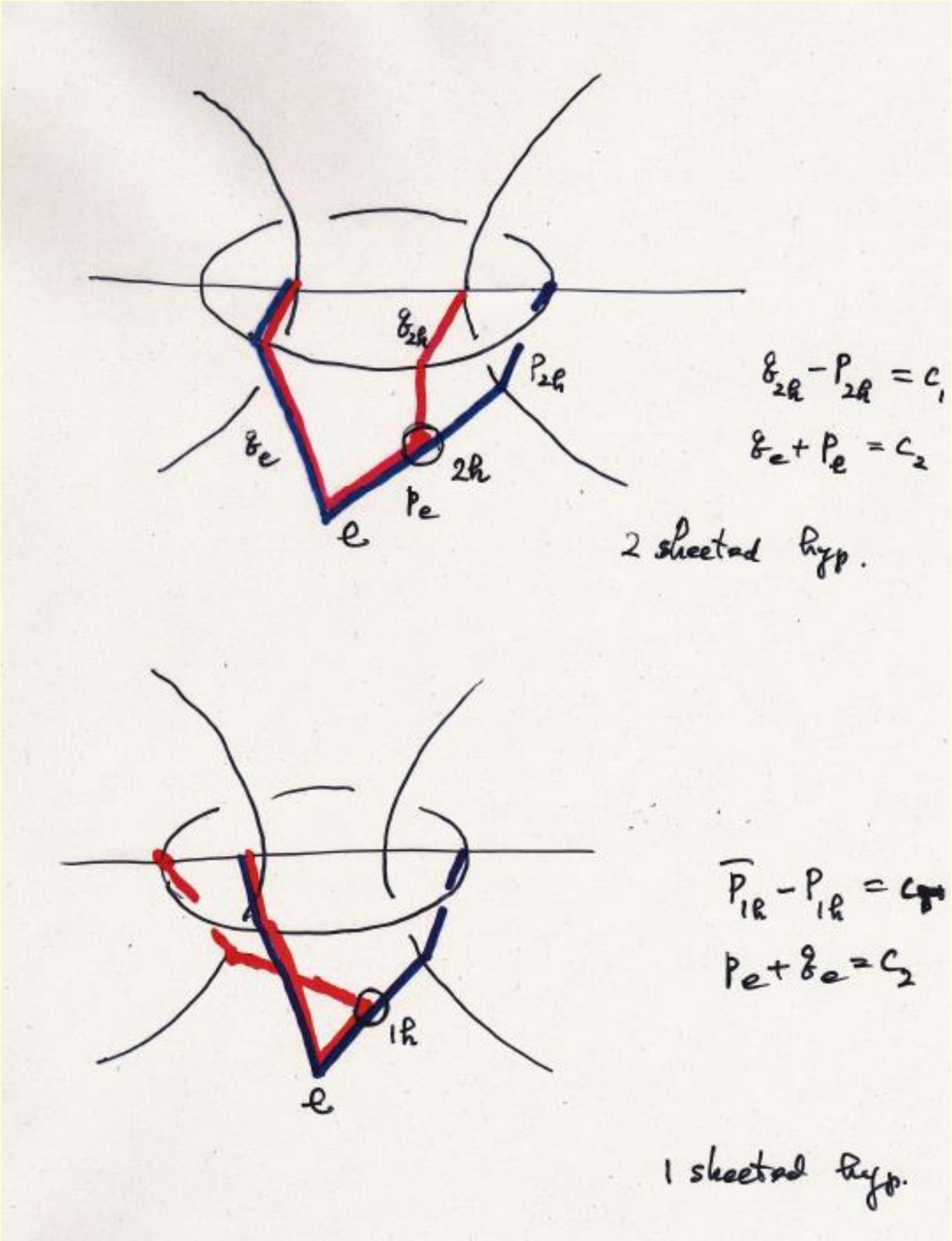


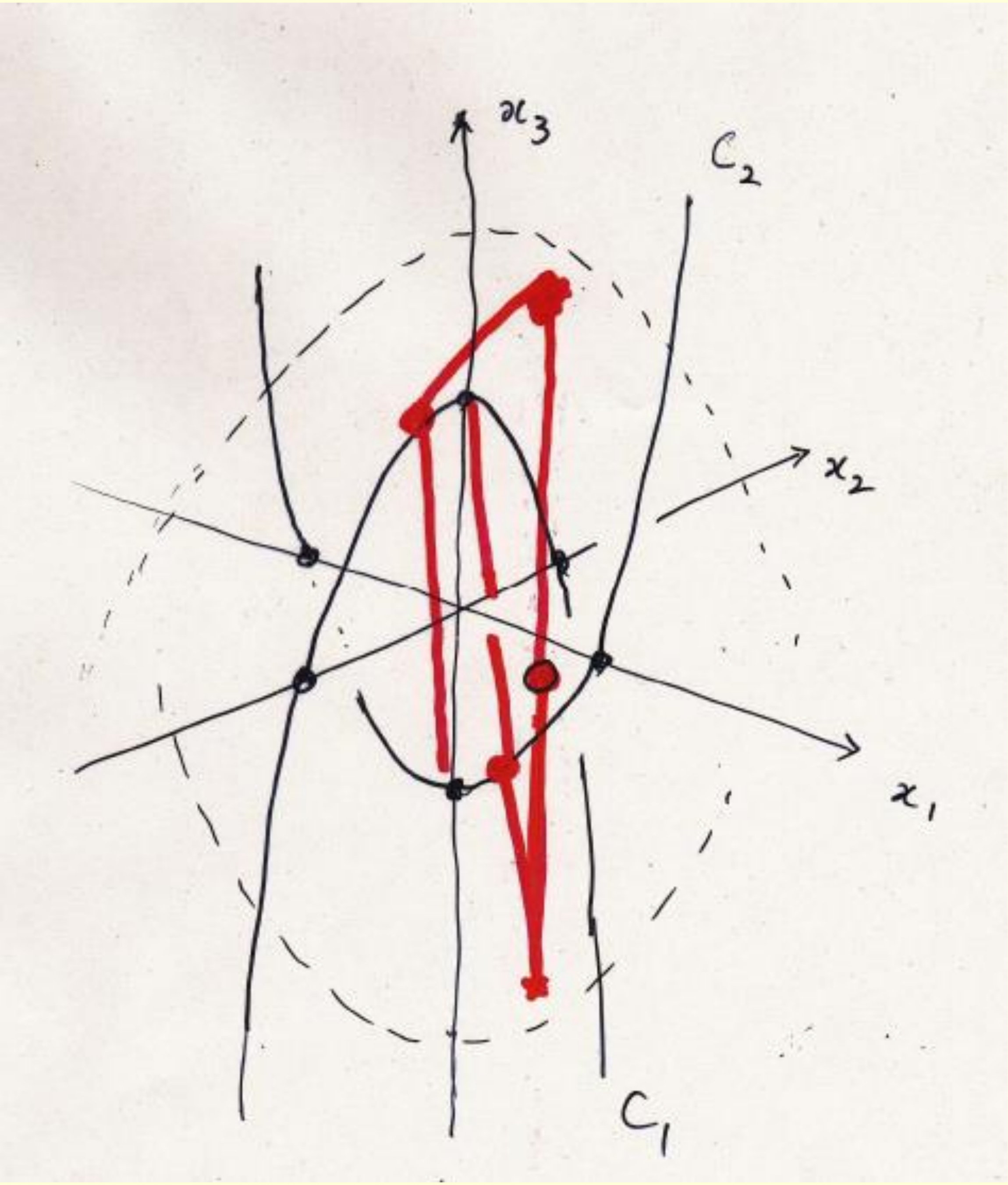
C_1^-
C_1^+
C_2
$\bar{p}$
$\bar{q}$
q
p
P
x_1


C_1^-
C_1^+
C_2
p
x_1



-r3(p)
-r2(p)
r1(p)
r2(p)
r3(p)
x0
-S2
-S1
S1
S2


q2h
p2h
qe
2h
pe
e
q2h − p2h = c1
qe + pe = c2
2 sheeted hyp.
1h
e
p̄1h − p1h =
pe + qe = c2
1 sheeted hyp.



x_3
C_2
x_2
x_1
C_1