Steinberg modules, Lecture 1

Jeremy Miller (Purdue)

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Goals

Goal: Study $H^i(\Gamma)$ for $\Gamma \leq \mathrm{SL}_n(\mathbb{Z})$ finite index and i large.

Motivation: $H^i(\Gamma)$ is important in number theory, K-theory, manifold theory, ...

Outline

- Review: Definition of $H_i(G; M)$ and $H^i(G; M)$.
- Classical facts about cohomology of finite index subgroups of $\mathrm{SL}_n(\mathbb{Z})$.
- Borel–Serre duality.
- Tits buildings.
- Steinberg modules.

$K(G,1)$'s (part 1)

Definition

Let G be a group. X is $K(G,1)$ if $\pi_1(X) = G$ and the universal cover is contractible (all other homotopy groups vanish).

Example

- $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2,1)$.
- $\vee_n S^1$ is a $K(F_n,1)$.
- $(S^1)^n$ is a $K(\mathbb{Z}^n,1)$.

$K(G, 1)$'s (part 2)

Definition

Let G be a group. X is $K(G, 1)$ if $\pi_1(X) = G$ and the universal cover is contractible (all other homotopy groups vanish).

Theorem

If X and Y are both $K(G, 1)$'s, then $X \simeq Y$.

Group homology

Theorem

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Definition

Let G be a group. Let $H_i(G) := H_i(K(G,1))$ and $H^i(G) := H^i(K(G,1))$.

Twisted coefficients

Proposition

A $\mathbb{Z}[G]$-module is the same data as a bundle of abelian groups over $K(G,1)$.

Definition

Let G be a group and M a $\mathbb{Z}[G]$-module. Let $H_i(G;M) := H_i(K(G,1);M)$ and $H^i(G) := H^i(K(G,1);M)$.

Basic facts

Proposition

$H_1(G) = G^{ab}$.

Proposition

$H^*(G)$ are characteristic classes for covers with automorphism group G.

Algebraic definition

Theorem

Let G be a group and M a $\mathbb{Z}[G]$-module. Then $H_i(G;M) \cong \mathrm{Tor}_i^{\mathbb{Z}[G]}(\mathbb{Z}, M)$ and $H^i(G;M) \cong \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, M)$.

Stable homology

Conventions: All homology will be with $\mathbb{Q}$ coefficients, $\Gamma \leq \mathsf{SL}_n(\mathbb{Z})$ finite index.

Theorem (Borel, Sun–Li)

For $* \leq n-2$, $H^*(\Gamma) \cong \bigwedge(x_5, x_9, x_{13}, \ldots)$.

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Surprising corollary:

Theorem (Farrell–Hsiang)

For $d \geq 5$ *and odd and* $i \leq d/3$, $\pi_i(\mathit{Diff}_\partial(D^d)) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q}, & \text{if } 4 \mid i, \\ 0 & \text{otherwise.} \end{cases}$

Unstable homology

Theorem (Borel, Sun–Li)

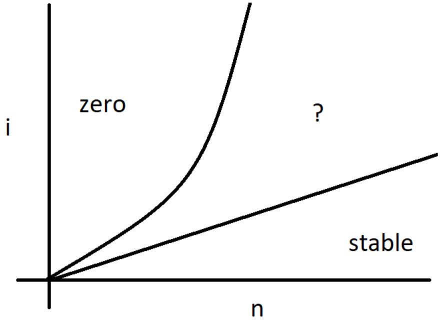
For $* \leq n-2$*,* $H^*(\Gamma) \cong \bigwedge(x_5, x_9, x_{13}, \ldots)$*.*

Goal: Study $H^i(\Gamma)$ for $i > n-2$.

Vanishing in heigh degrees

Theorem (Borel-Serre)

For $i > \binom{n}{2}$, $H^i(\Gamma) \cong 0$.



Bieri–Eckmann duality

Definition

A group G is called a duality group of dimension d if there is a $\mathbb{Z}[G]$-module $\mathbb{D}$ with $H^{d-i}(G) = H_i(G; \mathbb{D})$.

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Example

G is fundamental group of a compact d-manifold with contractible universal cover and $\mathbb{D}$ is the orientation bundle.

Borel–Serre duality

Theorem (Borel–Serre)

Γ is a duality group of dimension $\binom{n}{2}$ with dualizing module the Stienberg module $\mathrm{St}_n(\mathbb{Q})$.

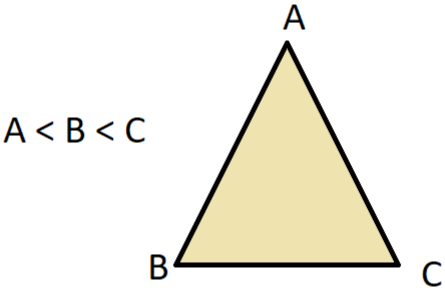
Since $H_i(\Gamma; \mathrm{St}_n(\mathbb{Q})) \cong 0$ for $i < 0$, $H^i(\Gamma) \cong 0$ for $i > \binom{n}{2}$.

Goal: Better understand $\mathrm{St}_n(\mathbb{Q}))$.

Tits building

Definition

Let F be a field and $T_n(F)$ be the simplicial complex with vertices subspaces $0 < V < F^n$ and $V_0, \ldots, V_p$ forming a p-simplex if $V_0 < V_1 < \ldots V_p$.



Solomon–Tits theorem (part 1)

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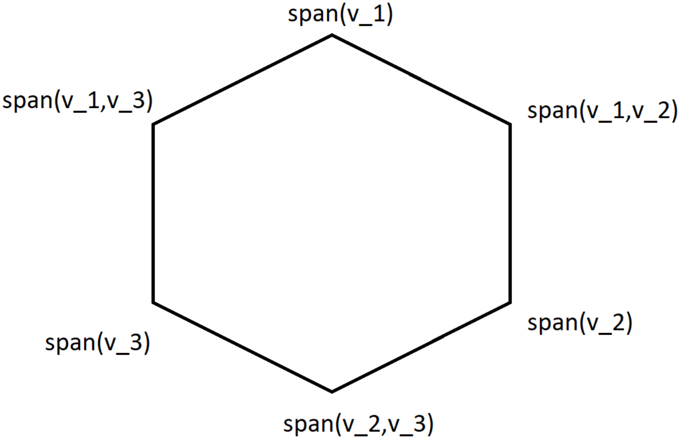
Definition (Solomon–Tits)

$St_n(F) := \widetilde{H}_{n-2}(T_n(F))$.

Spheres in $T_n(F)$

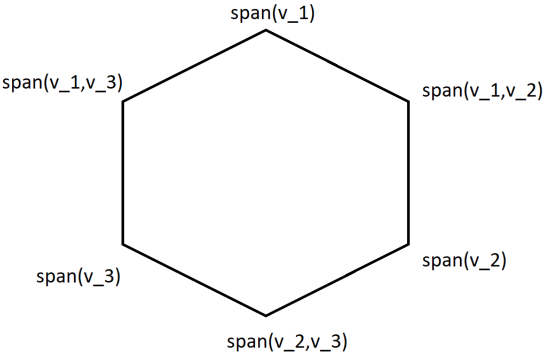
$T_3(F) \simeq \bigvee S^1$. Let v_1, v_2, v_3 be a basis of F^3.



Apartments

Definition

Let $\beta = \{v_1, \ldots, v_n\}$ be a basis of F^n. Let $S_\beta \cong S^{n-2}$ be the full subcomplex of $T_n(F)$ of spans of subsets of β (apartment). Let $[\beta]$ denote the image of $[S_\beta] \in \widetilde{H}_{n-2}(T_n(F)) = \mathrm{St}_n(F)$ (apartment class).



Solomon–Tits theorem (part 2)

Definition

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Theorem (Solomon–Tits theorem)

$\mathrm{St}_n(F)$ *is generated by apartment classes.*

Goals

Goal for next time: Find an even better generating set for $\mathrm{St}_n(\mathbb{Q})$ and use that to say something about $H^*(\Gamma)$.

Goal for today: Understand the Solomon–Tits theorem for $n = 3$.

Proof for $n = 3$ (part 1)

- $T_3(F)$ is a graph. Vertices are lines and planes in F^3. There is an edge from a plane to all the lines it contains.
- We need to first show $T_3(F) \simeq \bigvee S^1$ which is equivalent to showing it is connected.
- We will filter $T_3(F)$ by subspaces $X_0 \subset X_1 \ldots$ and inductively show X_i is connected.

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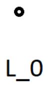
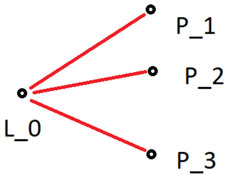
Fix a line $L_0 < F^3$. Let $X_0 = \{L_0\}$.



Proof for $n = 3$ (part 3)

We will filter $T_3(F)$ by subspaces $X_0 \subset X_1 \dots$ and inductively show X_i is connected.

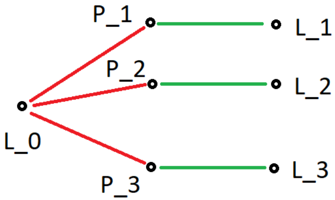
Fix a line $L_0 < F^3$. Let $X_0 = \{L_0\}$ and let X_1 full subcomplex on X_0 and planes containing L_0.



Proof for $n = 3$ (part 4)

We will filter $T_3(F)$ by subspaces $X_0 \subset X_1 \ldots$ and inductively show X_i is connected.

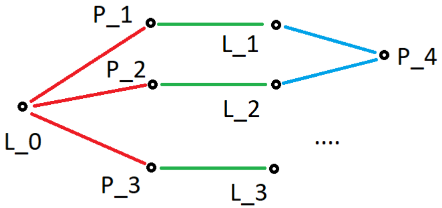
Fix a line $L_0 < F^3$. Let X_2 be full subcomplex on X_1 and lines.



Proof for $n = 3$ (part 5)

We will filter $T_3(F)$ by subspaces $X_0 \subset X_1 \ldots$ and inductively show X_i is connected.

Fix a line $L_0 < F^3$. Let $X_3 = T_3(F)$.



Proof for $n = 3$ (part 5)

We need to show $\tilde{H}_1(T_3(F))$ is generated by apartment classes. Since X_2 is contractible, $\tilde{H}_1(T_3(F))$ is generated by loops passing through exactly one vertex of X_3. These are apartments.