Steinberg modules, Lecture 2

Jeremy Miller (Purdue)

9/7/2023

Goals

Goal: Study $H^i(\Gamma)$ for $\Gamma \leq \mathrm{SL}_n(\mathbb{Z})$ finite index and i large.

Today we primarily will focus on the $H^{\binom{n}{2}}(\mathrm{SL}_n(\mathbb{Z}))$.

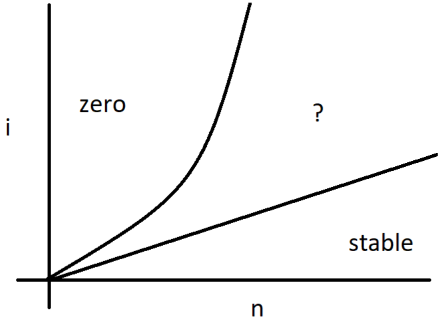
Outline

- Review of Lecture 1.
- Cohomology growth for high index subgroups.
- Church–Farb–Putman conjecture.
- Integral apartments.

Stable and unstable cohomology of $H^*(\Gamma)$

Let $\Gamma \leq SL_n(\mathbb{Z})$ finite index.

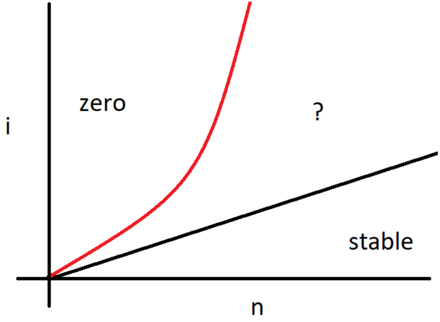
- $H^i(\Gamma)$ known if $i < n - 1$.
- $H^i(\Gamma) = 0$ if $i > \binom{n}{2}$.



Cohomology in degree $\binom{n}{2}$

Theorem (?????)

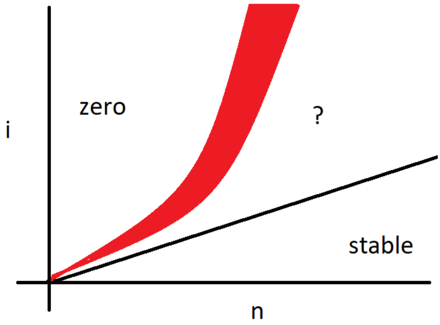
For all N and $n > 1$, there exist $\Gamma \leq SL_n(\mathbb{Z})$ finite index with $\dim H^{\binom{n}{2}}(\Gamma) > N$.



Church–Farb–Putman conjecture

Conjecture (Church–Farb–Putman)

$H^{\binom{n}{2}-i}(\mathrm{SL}_n(\mathbb{Z})) = 0$ *for* $i \leq n-2$.



Church–Farb–Putman conjecture (known results)

Conjecture (Church–Farb–Putman)

$H^{\binom{n}{2}-i}(\mathrm{SL}_n(\mathbb{Z})) = 0$ *for* $i \leq n-2$.

Conjecture true for small i:

- $i = 0$ due to Lee–Szczarba.
- $i = 1$ due to Church–Putman.
- $i = 2$ due to Bruck–M.–Patzt–Sorka-Wilson.

Borel–Serre duality

Theorem (Borel–Serre)

Γ is a duality group of dimension $\binom{n}{2}$ with dualizing module the Stienberg module $\mathrm{St}_n(\mathbb{Q})$.

Conjecture (Church–Farb–Putman (rephrased))

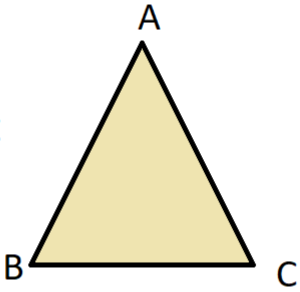
$H_i(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}_n(\mathbb{Q})) = 0$ *for* $i \leq n - 2$.

Tits building

Definition

Let F be a field and $T_n(F)$ be the simplicial complex with vertices subspaces $0 < V < F^n$ and $V_0, \ldots, V_p$ forming a p-simplex if $V_0 < V_1 < \ldots V_p$.

A < B < C



Solomon–Tits theorem (part 1)

Theorem (Solomon–Tits)

$T_n(F) \simeq \bigvee S^{n-2}$.

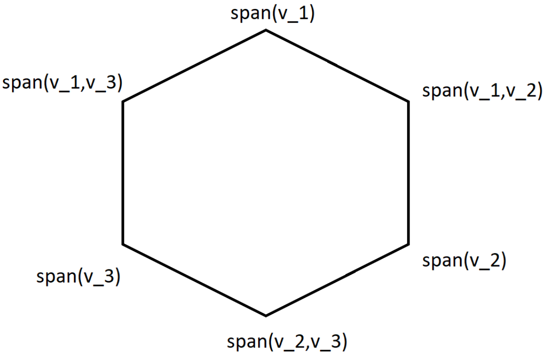
Definition (Solomon–Tits)

$St_n(F) := \widetilde{H}_{n-2}(T_n(F))$.

Apartments

Definition

Let $\beta = \{v_1, \ldots, v_n\}$ be a basis of F^n. Let $S_\beta \cong S^{n-2}$ be the full subcomplex of $T_n(F)$ of spans of subsets of β (apartment). Let $[\beta]$ denote the image of $[S_\beta] \in \widetilde{H}_{n-2}(T_n(F)) = \mathrm{St}_n(F)$ (apartment class).



Solomon–Tits theorem (part 2)

Definition

Let $\beta = \{v_1, \ldots, v_n\}$ be a basis of F^n. Let $S_\beta \cong S^{n-2}$ be the full subcomplex of $T_n(F)$ of spans of subsets of β (apartment). Let $[\beta]$ denote the image of $[S_\beta] \in \widetilde{H}_{n-2}(T_n(F)) = \mathrm{St}_n(F)$ (apartment class).

Theorem (Solomon–Tits theorem)

$\mathrm{St}_n(F)$ *is generated by integral apartment classes.*

Goals

Goal for today: Find an even better generating set for $\mathrm{St}_n(\mathbb{Q})$ and use that to prove the Church–Farb–Putman conjecture for $i = 0$.

Integral apartments

Definition

An apartment $S_\beta \subset T_n(\mathbb{Q})$ is integral $\beta = \{v_1, \ldots, v_n\}$ with $v_1, \ldots, v_n$ a basis of $\mathbb{Z}^n$ (as opposed to $\mathbb{Q}^n$).

Example

$\beta = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is not integral. $\gamma = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ is integral.

Ash–Rudolph theorem

Definition

An apartment $S_\beta \subset T_n(\mathbb{Q})$ is integral $\beta = \{v_1, \ldots, v_n\}$ with $v_1, \ldots, v_n$ a basis of $\mathbb{Z}^n$ (as opposed to $\mathbb{Q}^n$).

Theorem (Ash–Rudolph)

$\mathrm{St}_n(\mathbb{Q})$ *is generated by integral apartment classes.*

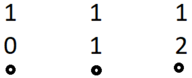
Ash–Rudolph theorem (example)

Theorem (Ash–Rudolph)

$\mathrm{St}_n(\mathbb{Q})$ *is generated by integral apartment classes.*

Example

$\beta = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is not integral. S_β is a sum of apartments associated to the matrices: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.



Coinvariants

Proposition

$H_0(G; M) = M/N$ with N generated by elements of the form $m - gm$.

Corollary

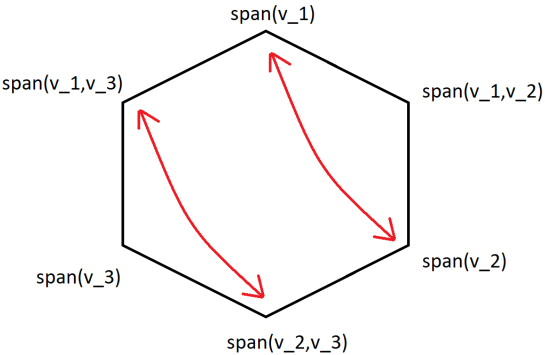
If M is a $\mathbb{Q}$-vector space and M is generated by elements m with the property that there exists g with $gm = -m$, then $H_0(G; M) = 0$.

Church–Farb–Putman conjecture for $i = 0$

- Want to show $H^{\binom{n}{2}}(\mathrm{SL}_n(\mathbb{Z})) = 0$.
- $H^{\binom{n}{2}}(\mathrm{SL}_n(\mathbb{Z})) \cong H_0(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}_n(\mathbb{Q}))$.
- $\mathrm{St}_n(\mathbb{Q})$ is generated by apartment classes S_β.
- For $\beta = \{v_1, \dots, v_n\}$, need to find $g \in \mathrm{SL}_n(\mathbb{Z})$ with $g(S_\beta) = -S_\beta$.
- Take $g(v_1) = v_2$, $g(v_2) = -v_1$, $g(v_i) = v_i$ for $i \geq 2$.

Church–Farb–Putman conjecture for $i = 0$ (with picture)

$g(v_1) = v_2$, $g(v_2) = -v_1$, $g(v_i) = v_i$ for $i \geq 2$. $g(S_\beta) = -S_\beta$.



Ash–Rudolph theorem again

Definition

An apartment $S_\beta \subset T_n(\mathbb{Q})$ is integral $\beta = \{v_1, \ldots, v_n\}$ with $v_1, \ldots, v_n$ a basis of $\mathbb{Z}^n$ (as opposed to $\mathbb{Q}^n$).

Still need to prove:

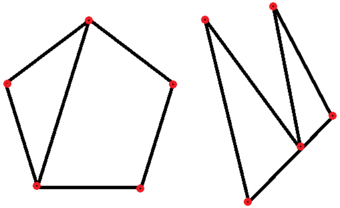
Theorem (Ash–Rudolph)

$\mathrm{St}_n(\mathbb{Q})$ *is generated by integral apartments.*

Elementary lemma

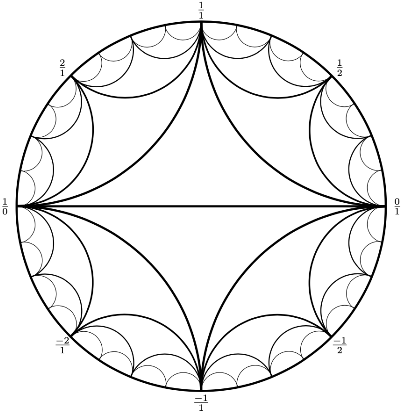
Lemma

Let G be a graph with vertex set V. Then $\tilde{H}_0(V)$ is generated differences of adjacent vertices iff G is connected.



Ash–Rudolph theorem $n = 2$

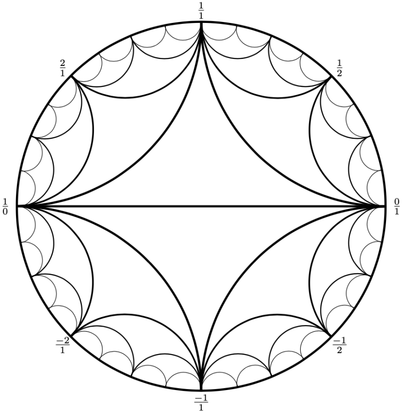
$T_2(\mathbb{Q})$ is the vertices of the Farey Graph. $\mathrm{St}_2(\mathbb{Q}) = \widetilde{H}_0(T_2(\mathbb{Q}))$. Integral apartments are the boundaries of edges in the Farey Graph. Generation is equivalent to connectivity of the Farey Graph.



Equivalent definition of the Farey Graph

Definition

Vertices of Farey graph are lines in $\mathbb{Q}^2$ (or equivalently rank 1 summands of $\mathbb{Z}^2$). Two summands L_1, L_2 form an edge if $\mathbb{Z}^2 = L_1 \oplus L_2$.



Path in the Farey Graph (part 1)

Start somewhere random: $\begin{bmatrix} 7 \\ 9 \end{bmatrix}$.

Goal: End up at $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Stategy lower the last coordinate.

Path in the Farey Graph (part 2)

Way to lower last last coordinates:
Given a vector $\vec{v}$, complete it to a basis $\vec{v}, \vec{w}$. If $\vec{w}$ has a larger last coordinate, substract $\vec{v}$ from it until the last coordinate is lower.

Path in the Farey Graph (part 2)

Way to lower last last coordinates:
Given a vector $\vec{v}$, complete it to a basis $\vec{v}, \vec{w}$. If $\vec{w}$ has a larger last coordinate, substract $\vec{v}$ from it until the last coordinate is lower.

Example: $\vec{v} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 17 \\ 22 \end{bmatrix}$. Edge from $\vec{v} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$ to $\vec{w} - 2\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Path in the Farey Graph (part 3)

Edge from $\begin{bmatrix} 7 \\ 9 \end{bmatrix}$ to $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Edge from $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Edge from $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Path in the Farey Graph (part 4)

Edge from $\begin{bmatrix}7\\9\end{bmatrix}$ to $\begin{bmatrix}3\\4\end{bmatrix}$. Edge from $\begin{bmatrix}3\\4\end{bmatrix}$ to $\begin{bmatrix}1\\1\end{bmatrix}$. Edge from $\begin{bmatrix}1\\1\end{bmatrix}$ to $\begin{bmatrix}1\\0\end{bmatrix}$.