Steinberg modules, Lecture 3

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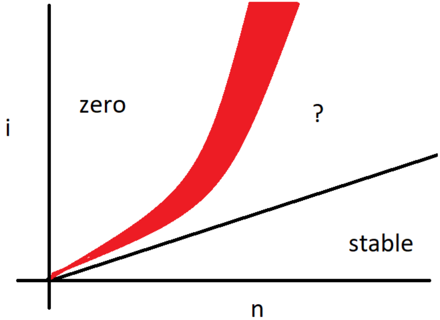
Outline

- Review
- Complex of partial bases
- Integral apartment map
- Map-of-poset lemma
- Presentations of Steinberg modules

Church–Farb–Putman conjecture

Conjecture

$H^{\binom{n}{2}-i}(\mathrm{SL}_n(\mathbb{Z})) = 0$ *for* $i \leq n-2$.



Ash–Rudolph theorem

Definition

An apartment $S_\beta \subset T_n(\mathbb{Q})$ is integral $\beta = \{v_1, \ldots, v_n\}$ with $v_1, \ldots, v_n$ a basis of $\mathbb{Z}^n$ (as opposed to $\mathbb{Q}^n$).

Theorem (Ash–Rudolph)

$\mathrm{St}_n(\mathbb{Q})$ *is generated by integral apartment classes.*

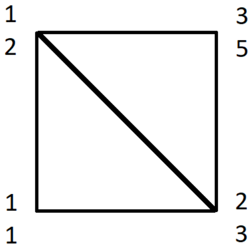
Last time: Ash–Rudolph implies Church–Farb–Putman conjecture for $i = 0$.

Goal: Topological proof of Ash–Rudolph (following Church–Farb–Putman).

Complex of partial bases

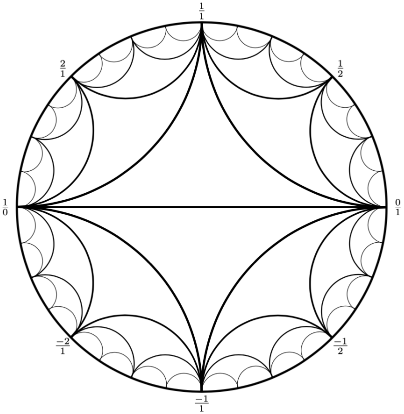
Definition

For R a ring, let $B_n(R)$ denote the complex with vertices unimodular vectors in R^n a collection of vertices forming a simplex if the vectors are a subset of a basis.



Complex of partial bases (n=2)

The difference between $B_2(\mathbb{Z})$ and the Farey graph is vertices are vectors instead of lines (twice as many vertices).



Span map

Let $B'_n(R)$ denote the subcomplex of $B_n(R)$ of simplices that are not bases.

Given a simplicial complex X, let $sd(X)$ denote the Barycentric subdivision.

Definition

Let $s : sd(B'_n(\mathbb{Z})) \to T_n(\mathbb{Q})$ be the map $\{v_0, \ldots, v_p\} \mapsto span(v_0, \ldots, v_p)$.

Integral apartment map

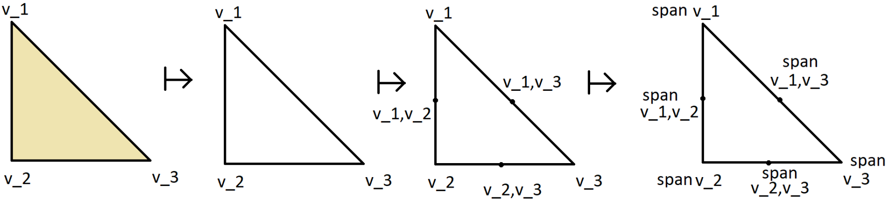
$$H_{n-1}(B_n(\mathbb{Z}), B'_n(\mathbb{Z})) \xrightarrow{\partial} \widetilde{H}_{n-2}(B'_n(\mathbb{Z})) \xrightarrow{s} \widetilde{H}_{n-2}(T_n(\mathbb{Q})).$$

$H_{n-1}(B_n(\mathbb{Z}), B'_n(\mathbb{Z})) = C_{n-1}(B_n(\mathbb{Z}))$,
free abelian group on the set of bases.

$H_{n-2}(T_n(\mathbb{Q})) = \mathrm{St}_n(\mathbb{Q})$.

Integral apartment map (picture)

$$C_{n-1}(B_n(\mathbb{Z})) = H_{n-1}(B_n(\mathbb{Z}), B'_n(\mathbb{Z})) \xrightarrow{\partial} \widetilde{H}_{n-2}(B'_n(\mathbb{Z})) \xrightarrow{s} \widetilde{H}_{n-2}(T_n(\mathbb{Q})) = \mathrm{St}_n(\mathbb{Q}).$$



Maazen's theorem

Theorem (Maazen)

$B_n(\mathbb{Z})$ is $(n-2)$-connected.

Corollary

$H_{n-1}(B_n(\mathbb{Z}), B_n(\mathbb{Z})') \to \widetilde{H}_{n-2}(B_n(\mathbb{Z})') \to \widetilde{H}_{n-2}(B_n(\mathbb{Z})) \cong 0$ is exact.

Thus, to show integral apartment map is surjective, just need to show $H_{n-2}(B_n'(\mathbb{Z})) \xrightarrow{s} \widetilde{H}_{n-2}(T_n(\mathbb{Q}))$ is surjective.

Poset (definition)

Definition

A poset is a set $\mathbb{A}$ and a relation $<$ such that for all $a, b, c \in P$:

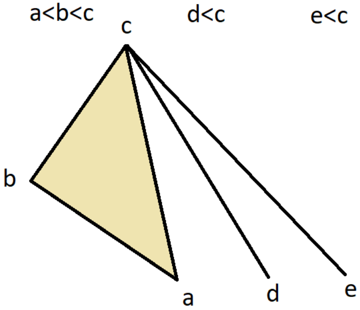
- $a \leq a$,
- $a \leq b$ and $b \leq a$ implies $a = b$,
- $a \leq b$ and $b \leq c$ implies $a \leq c$.

Write $a < b$ if $a \leq b$ and $a \neq b$.

Realization of Posets

Definition

If $\mathbb{A}$ is a poset, let $|\mathbb{A}|$ be the simplicial complex with vertices elements of P and with p simplices given by chains $a_0 < \ldots < a_p$.



Tits buildings via posets

Definition

Let $\mathbb{T}_n(F)$ denote the poset of subspaces $0 < V < F^n$ ordered by inclusion.

Proposition

$|\mathbb{T}_n(F)| \cong T_n(F)$.

Partial bases via posets

Definition

Let $\mathbb{B}_n(R)$ denote the poset of partial bases ordered by inclusion.

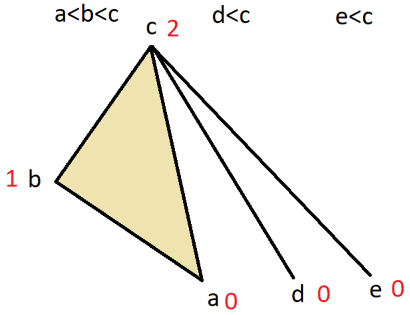
Proposition

$|B_n(R)| \cong sd(B_n(R))$.

Height

Definition

Let $a \in \mathbb{A}$. a has height $\geq k$ if there exists $a_0 < a_1 < a_2 < \cdots < a_k = a$.



Subsets of posets

Let $f : \mathbb{A} \to \mathbb{B}$ be order preserving, $b \in \mathbb{B}$.

Definition

Let $\mathbb{B}^{>b} = \{q \in \mathbb{B} \mid q > b\}$.

Definition

Let $f^{\leq b} = \{p \in \mathbb{A} \mid f(p) \leq b\}$.

Quillen lemma

Lemma (Quillen)

Let $f : \mathbb{A} \to \mathbb{B}$ be order preserving. Assume for all $b \in \mathbb{B}$, $\mathbb{B}^{>b}$ is $x - ht(b)$-connected and $|f^{\leq b}| : |\mathbb{A}| \to |\mathbb{B}|$ is $(y + ht(b))$-connected. Then f is $(x + y + 3)$-connected.

Goal: Apply this to $s : \mathbb{B}'_n(\mathbb{Z}) \to \mathbb{T}_n(\mathbb{Q})$.

$\mathbb{T}_n(\mathbb{Q})^{>V}$

Let $V \in \mathbb{T}_n(\mathbb{Q})$. $ht(V) = dim(V) - 1$.

$T_n(\mathbb{Q})^{>V} \cong T_{n-dim(V)}(\mathbb{Q}) \simeq \bigvee S^{n-dim(V)-2}$.

$T_n(\mathbb{Q})^{>V}$ is $(ht(V) + n - 4)$-connected.

Let $V \in \mathbb{T}_n(\mathbb{Q})$. $ht(V) = dim(V) - 1$.

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$s^{\leq V} \cong B_{dim(V)}(\mathbb{Z})$ and hence is $(dim(V) - 2) = (ht(V) - 1)$-connected.

Proof of Ash–Rudolph

Free abelian group on bases

$= H_{n-1}(B_n(\mathbb{Z}), B_n'(\mathbb{Z})) \xrightarrow{\partial} \widetilde{H}_{n-2}(B_n'(\mathbb{Z})) \xrightarrow{s} \widetilde{H}_{n-2}(T_n(\mathbb{Q})) = \mathrm{St}_n(\mathbb{Q})$.

- ∂ is surjective by Maazen's connectivity result.
- Quillen's poset lemma combined with Maazen's connectivity result and the Solomon–Tits Theorem imply s is $n-2$-connected and hence surjective on $\widetilde{H}_{n-2}$.

Church–Farb–Putman in codim 1

- Nice generators for $\mathrm{St}_n(\mathbb{Q})$ implies $H^{\binom{n}{2}}(\mathrm{SL}_n(\mathbb{Z})) \cong H_0(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}_n(\mathbb{Q})) = 0$ for $n \geq 2$.
- Nice presentation for $\mathrm{St}_n(\mathbb{Q})$ implies $H^{\binom{n}{2}-1}(\mathrm{SL}_n(\mathbb{Z})) \cong H_1(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}_n(\mathbb{Q})) = 0$ for $n \geq 3$.

Bykovskii presentation

Write $[[\vec{v}_1, \ldots, \vec{v}_n]]$= image of fundamental class of $S_{\vec{v}_1,\ldots,\vec{v}_n}$.

Theorem (Bykovskii)

$St_n(\mathbb{Q})$ is generated by symbols $[[\vec{v}_1, \ldots, \vec{v}_n]]$ with $\vec{v}_1, \ldots, \vec{v}_n$ a basis of $\mathbb{Z}^n$ subject to the following relations:

- $[[\vec{v}_1, \ldots, \vec{v}_n]] = sgn(\sigma)[[\vec{v}_{\sigma(1)}, \ldots, \vec{v}_{\sigma(n)}]]$
- $[[\vec{v}_1, \ldots, \vec{v}_n]] = [[\pm\vec{v}_1, \ldots, \pm\vec{v}_n]]$.
- $[[\vec{v}_1, \vec{v}_2 \ldots, \vec{v}_n]] - [[\vec{v}_1 + \vec{v}_2, \vec{v}_2 \ldots, \vec{v}_n]] + [[\vec{v}_1 + \vec{v}_2, \vec{v}_1 \ldots, \vec{v}_n]] = 0$.

Bykovskii presentation (n=2)

$$[[v_1, v_2]] - [[v_1 + v_2, v_2]] + [[v_1 + v_2, v_1]] = 0.$$