APPLYING CHAIN-LEVEL POINCARE DUALITY TO THE STRING TOPOLOGY OF THE 2-SPHERE

JOINT WITH THOMAS TRADLER

I - INTRO TO LOOP SPACE STRING TOPOLOGY

II - —"— ALGEBRAIC —"—

III - SOME ALGEBRAIC MACHINERY

IV - SOME CALCULATIONS FOR S^2; $\mathbb{Z}_2$

V - SPECULATION?

STATEMENTS

THM (2007 Menichi) $H^\cdot(LS^2;\mathbb{Z}_2)$ and $HH^\cdot(H^\cdot(S^2;\mathbb{Z}_2), H^\cdot(S^2;\mathbb{Z}_2))$ are not isomorphic as BV algebras

THM (2023 P-Tradler) Well, actually, we can "correct" the BV structure on $HH^\cdot$ so that there <u>is</u> an isomorphism of BV algebras.

DEF A BV algebra $(A,\cdot,\Delta)$ is a (graded) commutative associative algebra $(A,\cdot)$ together with an operator $\Delta: A_i \longrightarrow A_{i+1}$ such that

(1) $\Delta^2=0$

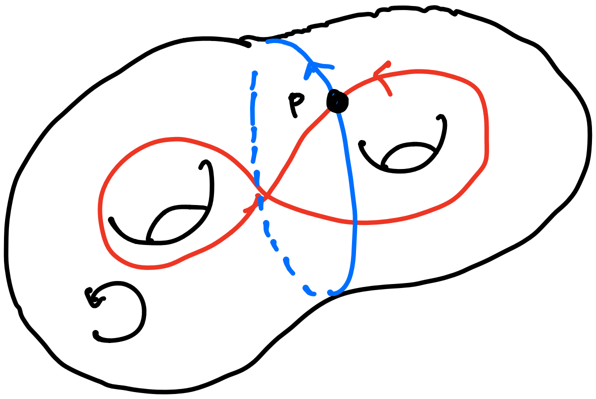
(2) Δ need not be a derivation of $\cdot$ but its deviation from being a derivation is a derivation in each variable

$$\Delta(a\cdot b)-(\pm\Delta(a)\cdot b \pm a\cdot\Delta(b)) = \{a,b\}$$

I STRING TOPOLOGY BV ALGEBRA

- study operations on loop spaces
- look for invariants of underlying spaces

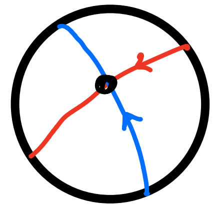
PRE HISTORY – GOLDMAN BRACKET



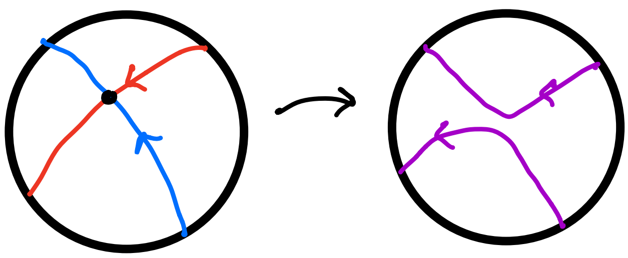
Σ closed oriented surface

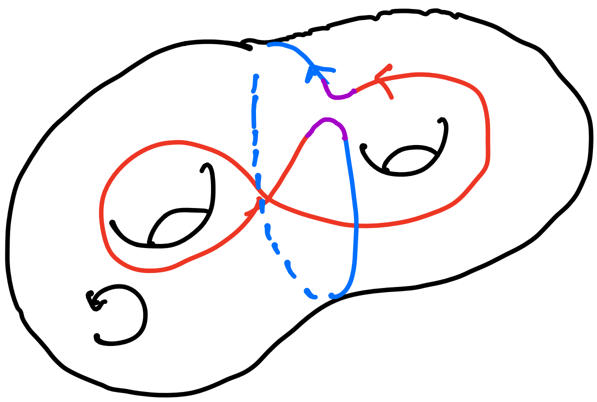
$\alpha, \beta : S^1 \to \Sigma$

intersect transversally at p



BASIC MOVE :





$\alpha \circ_p \beta$

DEF. Let π_0 be the set of free homotopy classes of loops on Σ

Let H be the free abelian group / R-module / $\mathbb{Q}$-vs generated by π_0

HUGE!

DEF (a) Let $[\alpha], [\beta] \in \pi_0$ then

$$\big[[\alpha],[\beta]\big] := \sum_{p \in \alpha \cap \beta} \pm [\alpha \cdot_p \beta]$$

(b) Extend linearly to

$$H \otimes H \xrightarrow{[\,,\,]} H$$: <u>Goldman bracket</u>

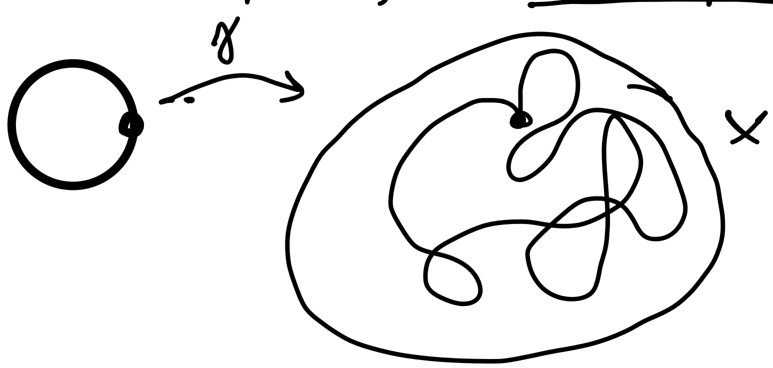
THM (Goldman, 80s) $[\,,\,]$ is well defined and gives H the structure of a Lie algebra

SLOGAN: String topology generalizes the Goldman bracket!

FIRST STRING TOPOLOGY OPERATIONS

DEF. X a space $LX := \mathrm{Maps}(S^1, X)$ <u>free loop space</u>

$S^1 = [0,1]/_{0\sim 1}$

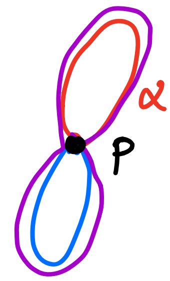


REMARK. If M is a manifold, LM can be given the structure of a manifold (∞-dim)

COMPARE Fix $p \in X$ $\Omega X := \mathrm{Maps}((S^1, 0), (X, p))$ <u>based loop space</u>

PRODUCT 1 ΩX has "concatenation" / "Pontryagin" product

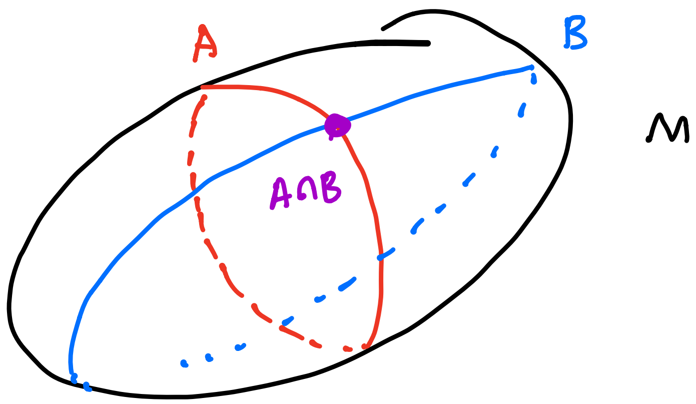
$$\Omega X \times \Omega X \xrightarrow{c} \Omega X$$



PRODUCT 2 M closed, oriented d-dim manifold

$$\begin{tikzcd}
H_i(M) \otimes H_j(M) \arrow[r, dashed, "\cap"] \arrow[d, "\sim"', "\text{inverse of PD iso}"] & H_{i+j-d}(M) \\
H^{d-i}(M) \otimes H^{d-j}(M) \arrow[r, "\cup"] & H^{2d-i-j}(M) \arrow[u, "\cap[M]"']
\end{tikzcd}$$

intersection product



CHAS-SULLIVAN LOOP PRODUCT - combine these 2 products

Let $\Delta^i \xrightarrow{\sigma} LM$ be an "oriented i-cell"

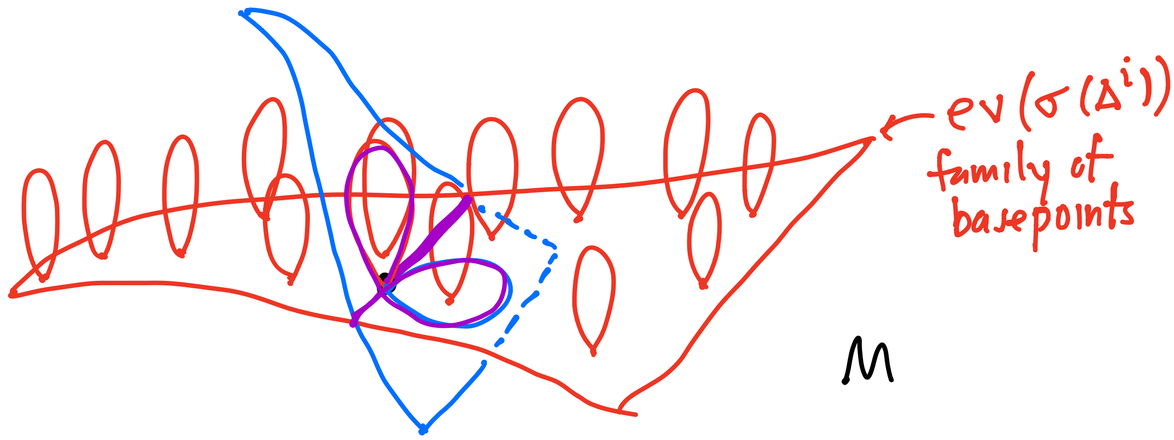
$$\updownarrow$$

$\Delta^i \times S^1 \longrightarrow M$ i-dimensional family of loops in M parametrized by Δ^i

Evaluation

$$\begin{array}{ccc} LM & & \gamma: S^1 \to M \\ \big\downarrow ev & & \big\downarrow \\ M & & \gamma(0) \end{array}$$

evaluate a loop at a basepoint



Let $\Delta^j \xrightarrow{\tau} LM$ be an "oriented j-cell in LM"

Assume families of basepoints intersect transversally in M

If p is in the intersection locus, have 2 loops with basepoint p

★ CONCATENATE ALONG INTERSECTION LOCUS ★

THM (Chas-Sullivan 99) This "parametrized concatenation" induces a "partially defined"

$$C_i(LM) \text{ "}\otimes\text{" } C_j(LM) \longrightarrow C_{i+j-d}(LM)$$

induces loop product

$$H_i(LM) \otimes H_j(LM) \xrightarrow{\bullet} H_{i+j-d}(LM)$$

(graded) associative, commutative

FURTHER

(a) $M \hookrightarrow LM$ constant loops induces

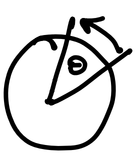
$$(H_\bullet(M), \cap) \longrightarrow (H_\bullet(LM), \bullet) \quad \text{alg morphism}$$

(b)
$$\begin{tikzcd} \Omega M \arrow[r] & LM \arrow[l, dashed, bend right] \arrow[d, "ev"] \\ & M \end{tikzcd}$$
restriction to base point induces

$$(H_\bullet(LM), \bullet) \longrightarrow (H_{\bullet-d}(\Omega M), c_*)$$

BV OPERATOR X any space

$$S^1 \times LX \xrightarrow{r} LX \quad \text{rotation}$$



$$(\theta, \gamma) \longmapsto (t \mapsto \gamma(t+\theta))$$

$$\rightsquigarrow H_i(S^1) \otimes H_j(LX) \xrightarrow{EZ} H_{i+j}(S^1 \times LX) \xrightarrow{r_\bullet} H_{i+j}(LX)$$

Fix $[S^1] \in H_1(S^1)$

induces $H_\bullet(LX) \xrightarrow{\Delta} H_{\bullet+1}(LX)$

THM (Chas-Sullivan)

$(H_\bullet(LM), \bullet, \Delta)$ is a BV algebra

THM (Cohen-Jones-Yan 2003, Menichi 2007)

$$H_\bullet(LS^2;\mathbb{Z}_2)[-2] \simeq \bigoplus_{k\geq 0} \mathbb{Z}_2[\alpha_k] \oplus \bigoplus_{k\geq 0} \mathbb{Z}_2[\beta_k]$$

$|\alpha_k| = k$ $\quad$ $|\beta_k| = k-2$

$$\alpha_k \cdot \alpha_l = \alpha_{k+l}$$
$$\beta_k \cdot \beta_l = 0$$
$$\alpha_k \cdot \beta_l = \beta_{k+l}$$

$$\Delta(\alpha_k) = 0$$
$$\Delta(\beta_k) = k\alpha_{k-1} + k\beta_{k+1}$$

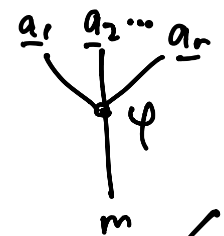
II HOCHSCHILD COHOMOLOGY BY ALGEBRA

Let A be a dga over a commutative ring R
M dg module over A

DEF. Hochschild cochain complex

shift $\underline{A}_j := A_{j-1}$

$$CH^\bullet(A,M) := \prod_{r\geq 0} \mathrm{Hom}(\underline{A}^{\otimes r}, M)$$



$$\varphi : \underline{A}^{\otimes r} \to M$$

Differential $D = D_d + D_\circ$

$d_{\mathrm{Hom}}(\varphi)$

$$D_d(\varphi)(\underline{a}_1 \ldots \underline{a}_r) := d_M \varphi(\underline{a}_1 \ldots \underline{a}_r) + \sum_j \pm \varphi(\underline{a}_1, \ldots, d\underline{a}_j \ldots \underline{a}_r)$$

$$D_\circ(\varphi)(\underline{a}_1, \ldots, \underline{a}_{r+1}) := a_1 \varphi(\underline{a}_2 \ldots \underline{a}_{r+1}) + \sum_j \pm \varphi(a_1 \ldots \underline{a_j a_{j+1}} \ldots \underline{a}_r)$$
$$\pm \varphi(\underline{a}_1 \ldots \underline{a}_r) \cdot a_{r+1}$$

$D^2 = 0$

<u>Hochschild cohomology</u>

$$HH^\bullet(A,M) := H^\bullet(CH^\bullet(A,M), D)$$

<u>normalized</u> cochains φ vanish if any input is $\underline{1} \in \underline{A}$

THM (Loday?) $\overline{CH}^\bullet(A,M) \hookrightarrow CH^\bullet(A,M)$ quasi iso

GERSTENHABER CUP PRODUCT ON $HH^\bullet(A,M)$

DEF $\varphi : \underline{A}^{\otimes r} \to A$ $\quad \varphi \cup \psi : \underline{A}^{\otimes(r+s)} \to A$

$\psi : \underline{A}^{\otimes s} \to A$ $\quad (\underline{a_1} \ldots \underline{a_{r+s}}) \mapsto \varphi(\underline{a_1} \ldots \underline{a_r}) \overset{m\,A}{\cdot} \psi(\underline{a_{r+1}} \ldots \underline{a_{r+s}})$

THM (Gerstenhaber 60s?) D is a derivation of $\cup$, induces

$$HH^\bullet(A,A) \otimes H^\bullet(A,A) \xrightarrow{\cup} H^\bullet(A,A)$$

THM (Cohen-Jones 2002) M closed, oriented, simply connected. Let $A = C^\bullet(M)$ singular cochains. $\exists$ isomorphism of graded commutative algebras

$$(H_\bullet(LM), \bullet) \longrightarrow (HH^\bullet(A,A), \cup)$$

QUESTION: What about a BV structure on $HH^\bullet(A,A)$?