APPLYING CHAIN-LEVEL POINCARE DUALITY TO THE STRING TOPOLOGY OF THE 2-SPHERE

JOINT WITH THOMAS TRADLER

- I - INTRO TO LOOP SPACE STRING TOPOLOGY
- II - —— " —— ALGEBRAIC —— " ——
- almost! III - SOME ALGEBRAIC MACHINERY
- IV - SOME CALCULATIONS FOR S^2; $\mathbb{Z}_2$
- V - SPECULATION?

THM (2023 P-Tradler)
There is an isomorphism of BV algebras. $\bullet, \Delta$

$$\underset{\cup,\ \Delta^F}{HH^\cdot(H^\cdot(S^2;\mathbb{Z}_2), H^\cdot(S^2;\mathbb{Z}_2))} \longrightarrow H_\cdot(LS^2;\mathbb{Z}_2)$$

where $F: H^\cdot(S^2) \xrightarrow{\cap [S^2]} H_\cdot(S^2)$ fundamental class
is the usual Poincaré duality isom.
(map of bimodules over $H^\cdot(S^2)$)

GOAL: $\phi_k, \psi_k, \cup, \cancel{\Delta^F}\ \Delta^F$ $\qquad\qquad$ α_k, β_k

$$\begin{tikzcd}
HH^\cdot(H^\cdot(S^2;\mathbb{Z}_2), H^\cdot(S^2;\mathbb{Z}_2)) \arrow[r, "\Theta"] \arrow[d, "F"'] \arrow[d, shift left=3, "\tilde{F}"] & H_\cdot(LS^2;\mathbb{Z}_2) \\
HH^\cdot(H^\cdot(S^2;\mathbb{Z}_2), H_\cdot(S^2;\mathbb{Z}_2)) &
\end{tikzcd}$$

$\bullet, \Delta$

θ_k, ψ_k, β

★ want a better $F: H^\cdot \xrightarrow{\sim} H_\cdot$
so Θ is isom of BV alg ★

RECALL

F is induced by $C^\bullet(S^2) \xrightarrow{\cap [S^2]} C_\bullet(S^2)$ fundamental cycle

which is a right-module map but not a left module map in general.

BUT it is a left-module map "up to homotopy"

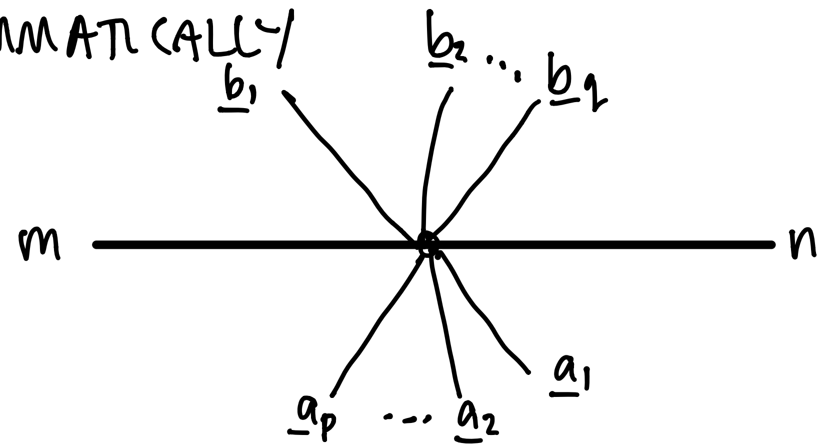
BIMODULE MAPS "UP TO HOMOTOPY"

$(A, d, \cdot)$ is a dga, M dg bimod over A.

Consider a collection

$$F = \left\{ F_{p,q} : \underline{A}^{\otimes p} \otimes M \otimes \underline{A}^{\otimes q} \longrightarrow M^* \right\}$$

DIAGRAMMATICALLY



Define $DF = \{(DF)_{p,q}\}$ by

$$(DF)_{p,q} = (D_d F)_{p,q} + (D_\cdot F)_{p,q}$$

analogous to Hochschild differential

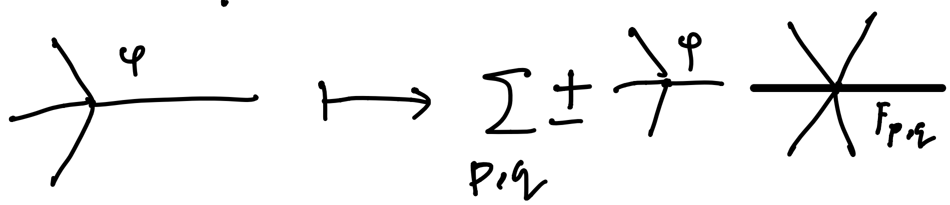
ABUSE OF NOTATION: $F: M \longrightarrow M^*$

DEF. The collection F is called a <u>homotopy inner product</u> if $DF = 0$

NOTICE: If $F_{p,q} = 0$ for $(p,q) \neq (0,0)$ then $F_{0,0}: A \longrightarrow A^*$ is a bimodule map (AKA a regular inner product)

DEF. F induces

$$CH^\bullet(A,A) \xrightarrow{\mathcal{F}} CH^\bullet(A,A^*)$$



$$\mathrm{Hom}(\underline{A}^{\otimes r}, A) \xrightarrow{\text{each compoment}} \mathrm{Hom}(\underline{A}^{\otimes r+p+q}, A^*)$$

REMARK If $DF = 0$ then $\mathcal{F}$ is a chain map, induces

$$\begin{array}{c} HH^\bullet(A,A) \\ \downarrow \mathcal{F} \\ HH^\bullet(A,A^*) \end{array}$$

EXAMPLE: $A = H^\bullet(S^2; \mathbb{Z}/2)$ $\quad F = \{F_{0,0}: A \xrightarrow{\cap [S^2]} A^*\}$ usual poincaré duality — induces $\mathcal{F}$ from before

IV - SOME CALCULATIONS FOR $S^2; \mathbb{Z}_2$

REMINDER

GOAL: $\phi_k, \psi_k, \cup, \cancel{\Delta^F}\ \Delta^F$ $\qquad\qquad$ α_k, β_k

$$\begin{tikzcd} HH^\bullet(H^\bullet(S^2;\mathbb{Z}_2), H^\bullet(S^2;\mathbb{Z}_2)) \arrow[r, "\Theta"] \arrow[d, shift right, "\mathcal{F}"'] \arrow[d, shift left, "\tilde{\mathcal{F}}"] & H_\bullet(LS^2;\mathbb{Z}_2) \\ HH^\bullet(H^\bullet(S^2;\mathbb{Z}_2), H_\bullet(S^2;\mathbb{Z}_2)) & \end{tikzcd}$$

$\bullet, \Delta$

θ_k, ψ_k, β

★ want a better $F: H^\bullet \xrightarrow{\sim} H_\bullet$ so Θ is isom of BV alg ★

$A = H^\bullet := H^{-\bullet}(S^2;\mathbb{Z}_2)$

generated by e, s $\quad |e| = 0 \quad |s| = -2$

$HH^\bullet(H^\bullet, H^\bullet)$ generated by $\phi_k, \psi_k \quad k \geq 0$

$\phi_k(\underline{s}, \dots, \underline{s}) = e \qquad \psi_k(\underline{s}, \dots, \underline{s}) = s$

$|\phi_k| = k \qquad\qquad |\psi_k| = k-2$

$\phi_k \cup \phi_\ell = \phi_{k+\ell}$

$\psi_k \cup \psi_\ell = 0$

$\phi_k \cup \psi_\ell = \psi_{k+\ell}$

$[S^2]$

$$A^* = H_\bullet = H_\bullet(S^2, \mathbb{Z}_2)$$

generated by e^*, s^*, $|e^*| = 0$, $|s^*| = 2$

$HH^\bullet(H^\bullet, H_\bullet)$ generated by θ_k, χ_k, $k \geq 0$

$$\theta_k(\underline{s}, \dots, \underline{s}) = s^* \qquad \chi_k(\underline{s}, \dots, \underline{s}) = e^*$$

$$|\theta_k| = k+2 \qquad |\chi_k| = k$$

$$\beta(\theta_k) = 0$$

$$\beta(\chi_k) = k\,\theta_{k-1}$$

usual Poincaré duality

$$F : H^\bullet \longrightarrow H_\bullet \qquad F(s) = e^*, \quad F(e) = s^*$$

induces

$$\begin{array}{c} HH^\bullet(H^\bullet; H^\bullet) \\ \text{iso} \downarrow \mathcal{F} \\ HH^\bullet(H^\bullet, H_\bullet) \end{array}$$

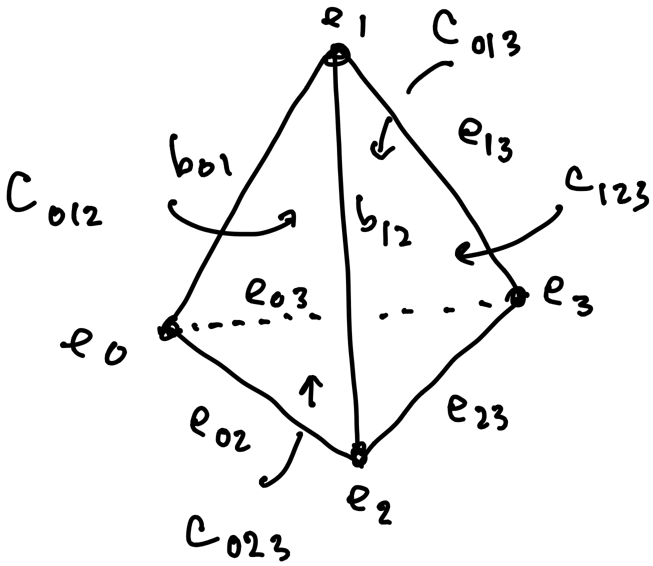
$$\boxed{\begin{array}{l} \mathcal{F}(\phi_k) = \theta_k \\ \mathcal{F}(\psi_k) = \chi_k \end{array}}$$

transfer β to $HH^\bullet(H^\bullet, H^\bullet)$

$$\Delta^F(\phi_k) = 0$$

$$\Delta^F(\psi_k) = k\,\phi_{k-1}$$

CHAIN LEVEL VERSION $A = C^{-\bullet}_{simp}(\partial\Delta^3; \mathbb{Z}_2)$



$D\tilde{F}^{ijk} = \tilde{F}^{ij} + \tilde{F}^{jk} + \tilde{F}^{ik}$

$\neq 0$

but $D\tilde{F} = 0$

IDEA $\tilde{F} : A \longrightarrow A^*$

$= \{\tilde{F}_{p,q}\}$

$= \tilde{F}^{012} + \tilde{F}^{023} + \tilde{F}^{013} + \tilde{F}^{123}$

where $\tilde{F}^{ijk} : C^{-\bullet}_{simp}(\Delta^2_{ijk}) \longrightarrow C^{simp}_{\bullet}(\Delta^2_{ijk})$

Actually even for Δ^2 this is complicated!

Let's define $\tilde{F}^{01}$ for Δ^1 instead...

<u>start with</u> $\tilde{F}^0$!

$A^{[0]} := C^{-\bullet}_{simp}(\Delta^0)$ gen by e_0

$\tilde{F}^{[0]} = \tilde{F}^{[0]}_{0,0} : A^{[0]} \longrightarrow (A^{[0]})^*$

$\tilde{F}^{[0]}(e_0)(e_0) = 1$

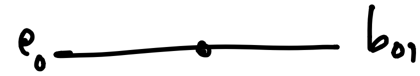
Back to Δ^1 ...

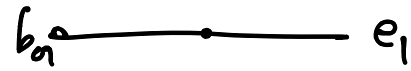
$A^{[01]} := C^{-\bullet}_{simp}(\Delta^1)$ (w usual $\cup$ product)

generated by $\underset{\substack{\uparrow \\ 0}}{e_0}, \underset{\substack{\uparrow \\ 0}}{e_1}, \underset{\substack{\uparrow \\ -1}}{b_{01}}$

$$\tilde{F}^{[01]}_{0,0} : A^{[01]} \longrightarrow \left(A^{[01]}\right)^*$$

nonzero only for $\begin{cases} \tilde{F}^{[01]}_{0,0}(e_0)(b_{01}) = 1 \\ \tilde{F}^{[01]}_{0,0}(b_{01})(e_1) = 1 \end{cases}$

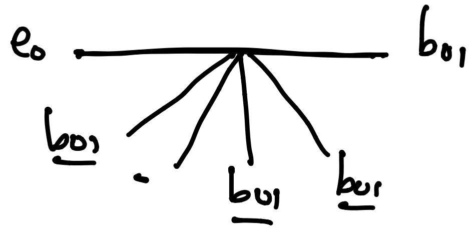


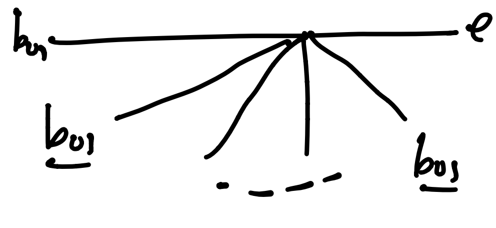


right module map but not a left module map

inductive procedure to define homotopies (previous work)

nonzero only for $\begin{cases} \tilde{F}^{[01]}_{k,0}(\underline{b_{01}}, \dots, \underline{b_{01}}; e_0)(b_{01}) = 1 \\ \tilde{F}^{[01]}_{k,0}(\underline{b_{01}}, \dots, \underline{b_{01}}; b_{01})(e_1) = 1 \end{cases}$



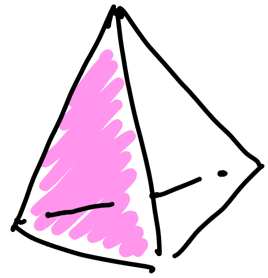


OBSERVATIONS

(1) $\tilde{F}^{[01]}_{0,0}$ is a map of bimodules up to the homotopy $\tilde{F}_{1,0}$

(2) $D\tilde{F}^{[01]} = \tilde{F}^{[0]} + \tilde{F}^{[1]}$

⋮
calculations
⋮



PROP Define

$$H^\bullet \downarrow \quad f : H^\bullet(S^2) \longrightarrow C^\bullet_{simp}(\partial \Delta^3)$$
$$e \longmapsto e_0 + e_1 + e_2 + e_3$$
$$s \longmapsto c_{012}$$

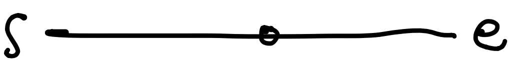
← choice

f is a quasi isomorphism

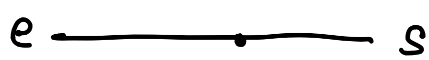
$f, \tilde{F}$ induce $\tilde{F} : H^{-\bullet}(S^2) \longrightarrow H_\bullet(S^2)$

↖ $\{\tilde{F}_{p,q}\}$

nonzero only for



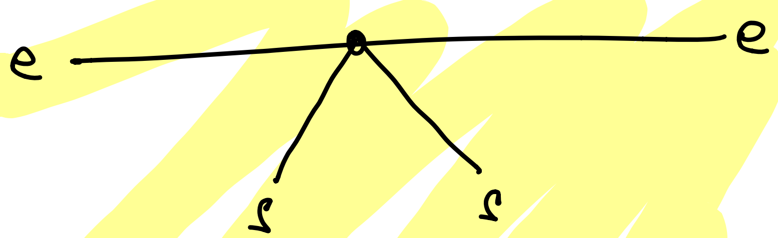
$\tilde{F}_{0,0}(s)(e) = 1$



$\tilde{F}_{0,0}(e)(s) = 1$

OLD - same as usual PD F

AND



$\tilde{F}_{2,0}(\underline{s}, \underline{s}, e)(e) = 1$

NEW!

This $\widetilde{F}$ induces

$$
\begin{array}{c}
\circlearrowleft \Delta^{\widetilde{F}} \\
HH^{\bullet}(H^{\bullet}, H^{\bullet}) \\
\Big\downarrow \widetilde{\mathcal{F}} \\
HH^{\bullet}(H^{\bullet}, H_{\bullet}) \\
\circlearrowright \\
\beta
\end{array}
$$

NEW!

$$
\boxed{
\begin{aligned}
\widetilde{\mathcal{F}}(\phi_k) &= \widetilde{F}_{00}(\phi_k) + \widetilde{F}_{2,0}(\phi_k) \\
&= \theta_k + \psi_{k+2} \\
\widetilde{\mathcal{F}}(\psi_k) &= \widetilde{F}_{0,0}(\psi_k) \\
&= \chi_k
\end{aligned}
}
$$

$$
\widetilde{\mathcal{F}}^{-1}(\theta_k) = \phi_k + \psi_{k+2}
$$

$$
\widetilde{\mathcal{F}}^{-1}(\chi_k) = \psi_k
$$

So

$$
\Delta^{\widetilde{F}}(\phi_k) = k\,(\phi_{k+1} + \psi_{k+3})
$$

$$
\Delta^{\widetilde{F}}(\psi_k) = k\,(\phi_{k-1} + \psi_{k+1})
$$

THM

$$
\begin{array}{ccc}
HH^{\bullet}(H^{\bullet}, H^{\bullet}) & \xrightarrow{\ \Theta\ } & H_{\bullet}(LS^2) \\
\cup,\ \Delta^{\widetilde{F}} & & \bullet,\ \Delta
\end{array}
$$

$$
\Theta(\phi_k) = \alpha_k + k\beta_{k+2}
$$

$$
\Theta(\psi_k) = \beta_k
$$

is an isomorphism of BV algebras!

VI SPECULATION

(1) What about for $M \neq S^2$??

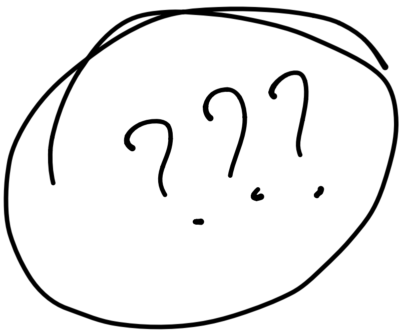
(2) Maybe can calculate <u>different</u> BV structures for lens spaces $L(7,1)$ and $L(7,2)$??

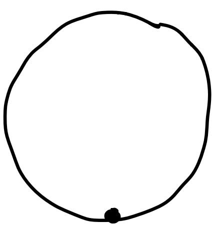
(3) Maybe just some nice pictures of low-dim classes of LS^2?

$$H_\cdot(LS^2, \mathbb{Z}) = \begin{cases} \mathbb{Z}\cdot\beta_0 \\ \mathbb{Z}\,\beta_1 \\ \mathbb{Z}_2\,\beta_2 \oplus \mathbb{Z}\,\alpha_0 \\ \mathbb{Z}\cdot\beta_3 \\ \mathbb{Z}_2\,\beta_4 \oplus \mathbb{Z}\,\alpha_2 \\ \mathbb{Z}\,\beta_5 \\ \vdots \end{cases}$$

Δ^F

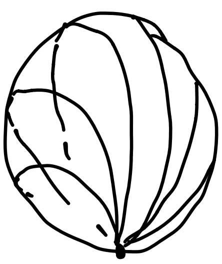
$\tilde{\Delta}^F$





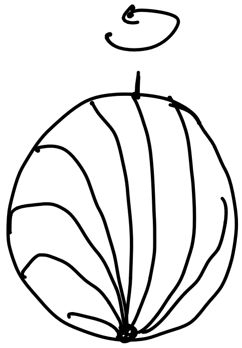
β_0

$2\beta_2 = 0$?

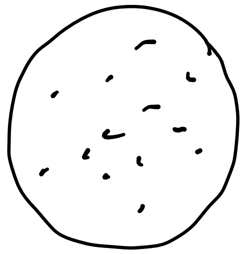


β_1

$$\tilde{\Delta}^F(\beta_1) = \beta_2 + \alpha_0$$



β_2



$\alpha_0 = [S^2]$