Classification of knots in $S_g \times S^1$ with small number of crossings

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Definition 1.1

A virtual link is an equivalence class of virtual knot diagrams modulo generalized Reidemeister moves. That is,

$$\{Virtual\ link\} = \{Virtual\ link\ diagrams\}/\langle moves\rangle$$

A virtual link, which has a representative with no virtual crossings, is called *a classical link.*

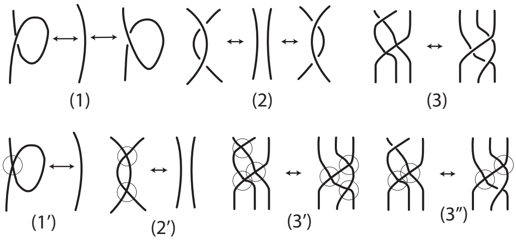


Figure 1: Generalized Reidemeister moves

Definition 1.2

A virtual link is a smooth embedding L of a disjoint union of S^1 into $S_g \times [0,1]$. Each image of S^1 is called *a component* of L. A link of one component is called *a virtual knot*.

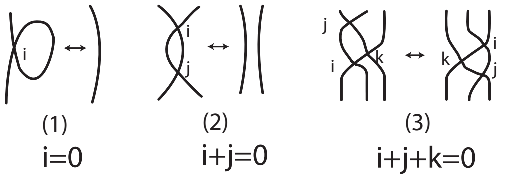
Definition 1.3

Let L and L' be two virtual links. If L' can be obtained from L by diffeomorshisms and stabil/destabilization of $S_g \times [0,1]$, then we call L and L' are *equivalent*.
A virtual link, which is equivalent to a link in $S^2 \times [0,1]$, is called *a classical links*.

Parity

Definition 1.4 (Gaußian parity)

We say that two chords of a formal Gauß diagrams are *linked* if their ends alternate. A chord is *(Gauß) odd* if it is linked with *oddly many* other chords; otherwise it is called *(Gauß) even*. Crossings of a virtual knot are named *odd* or *even* with respect to the parity of the corresponding chords.



Parity

- By using parity, invariants for classical knots can be extended to invariants for virtual knots.
- In particular, parity bracket is a picture-valued invariant.
- For virtual knots, the first homology of the underlying surface of a virtual knot can be used to obtain a parity.
- Parity is quite simple concept for virtual knots, but it provides plentiful information for virtual knots more than expected.

But,...

- Parity is defined just for virtual “knots”, not but for “links”.
- For classical knots, parity cannot give further information, because it is always trivial for classical knots.

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Motivation: Fibered knots and virtual knots

Definition 1.5

A fibered link in S^3 is a collection of disjointly embedded circles $L = L_1 \sqcup \cdots \sqcup L_k$ such that $S^3 - N(L)$ is diffeomorphic to the quotient of $F \times I$ by an equivalence $(x, 0) \sim (h(x), 1)$ where h is a diffeomorphism of F with $h|\partial F = Id$, where F is a Seifert surface of L.

In [2] M. Chrisman and V.O. Manturov studied virtual knots by using 2-component link $K \sqcup J$ with $lk(K, J) = 0$, where J is a fibered knot.

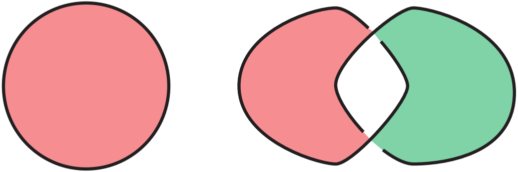
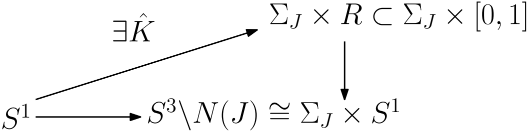


Figure 2: Seifert surfaces of a link

Let $K \sqcup J$ be a 2-component link where J is a fibered knot. Then

$$K \subset S^3 \setminus N(J) \cong \Sigma_J \times S^1.$$



If $lk(K, J) = 0$, then $\hat{K}$ is a knot in $\Sigma_J \times [0, 1]$ and it is a virtual knot.

But, if $lk(K, J) \neq 0$, then $\hat{K}$ is not closed.

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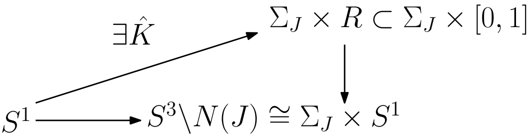

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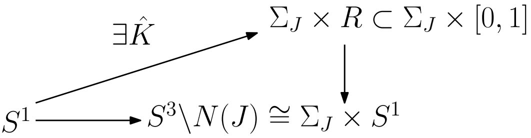

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But, if $lk(K, J) \neq 0$, then $\hat{K}$ is not closed.

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Definition 2.1

Let S_g be an oriented surface of genus g. *A link L in $S_g \times S^1$* is a pair of $S_g \times S^1$ and a smooth embedding of a disjoint union of S^1 into $S_g \times S^1$. We denote it by $(L, S_g \times S^1)$. Each image of S^1 in $S_g \times S^1$ is called *a component* of L. A link of one component is called *a knot in $S_g \times S^1$*.

Definition 2.2

Let $(L, S_g \times S^1)$ and $(L', S_{g'} \times S^1)$ be two links in $S_g \times S^1$. We call $(L, S_g \times S^1)$ and $(L', S_{g'} \times S^1)$ are *equivalent,* if $(L', S_{g'} \times S^1)$ can be obtained from $(L, S_g \times S^1)$ by isotopy and stabilization/destabilization of $S_g \times S^1$.

Roughly speaking, stabil/destabilization of $S_g \times S^1$ are addition/deletion of handles of S_g.

Diagrams for links in $S_g \times S^1$

Assume $x_0 \in S^1$ is a point such that $S_g \times \{x_0\} \cap L(S^1)$ is a set of finite points with no transversal points.

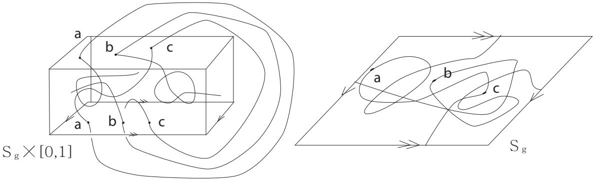

Figure 3:

Assume that counterclockwise orientation is given on S^1. Then $cl(S_g \times S^1) \setminus (S_g \times \{x_0\}) \cong_h S_g \times [0,1]$ where h is an orientation preserving diffeomorphism.

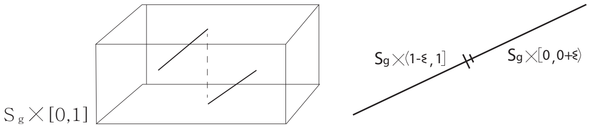

Figure 4: Change vertices to double lines

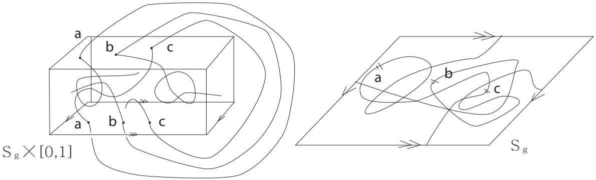

Figure 5: A diagram on S_g with double lines

A link L in M_L has a diagram *on the plane* with *virtual crossings* as virtual links. The following theorem also holds.

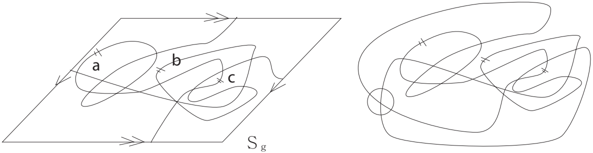

Figure 6: A diagram on the plane with double lines

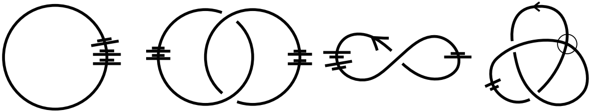

Figure 7: Examples of diagrams on the plane with double lines

Theorem 2.3 (K. [3] (2018))

Let L and L′ be two links in $S_g \times S^1$. Let D_L and $D_{L'}$ be diagrams of L and L′ on the plane, respectively. Then L and L′ are equivalent if and only if $D_{L'}$ can be obtained from D_L by applying the following moves in Fig. 8.

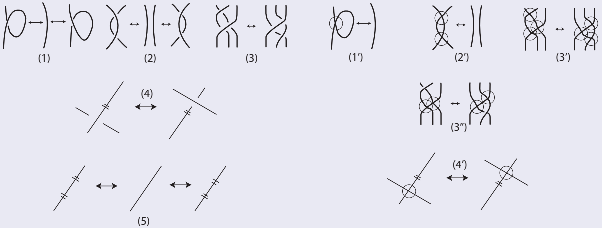

Figure 8: *Moves for links in $S_g \times S^1$*

In 2009 (almost) same theorem is proved by M. K. Dabkowski and M. Mroczkowski in [6] by using “arrows”.

Remark 2.4

By adding two arcs one can change over/under information of a crossing. That is, one can say that for links in $S_g \times S^1$ we miss over/under information.

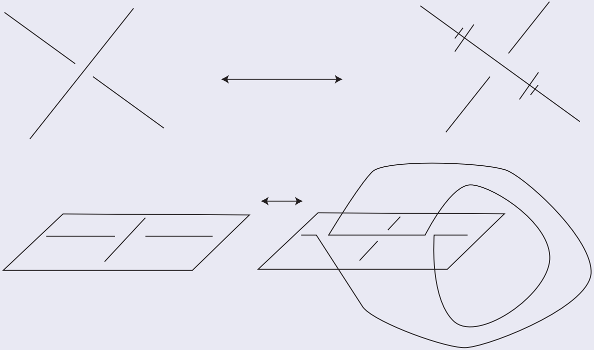

Figure 9: Crossing change with two additional double lines

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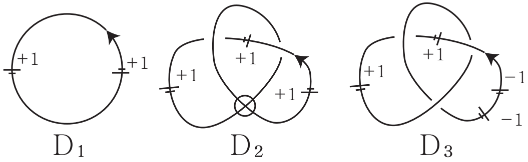
Degree of knots in $S_g \times S^1$

Let K be an oriented knot in $S_g \times S^1$ and D a diagram. For each double line, we give ± 1 with respect to the orientation.

The degree of K, deg(K) is the sum of ± 1's for all double lines of D.

$$deg(K) = p_*([K]) \in \mathbb{Z} = \pi_1(S^1)$$

where $p : S_g \times S^1 \to S^1$.



A label of a crossing

For a crossing c take a closed curve γ_c starting from under-crossing of c. *A label of c* valued in $\mathbb{Z}_{deg(K)}$ is the sum of ± 1's for double lines on γ_c.

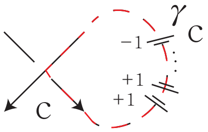
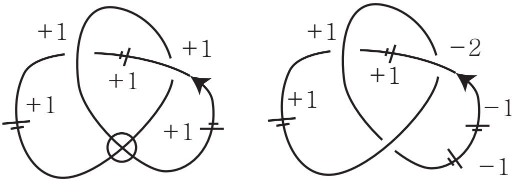
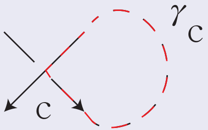


Figure 10: A label of a crossing



Remark 2.5

Geometrically, the label of a crossing c means how many times the curve from c to c turns around S^1.



Lemma 2.6

The labels for crossings satisfy the properties described in Fig. 11.

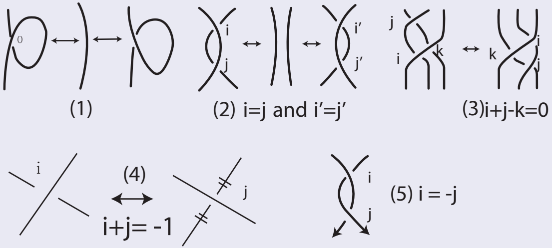


Figure 11:

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Definition 2.7

Let A be an abelian group. A *winding parity on diagrams of a knot* $\mathcal{K}$ *with coefficients in* A is a family of maps $p_K : \mathcal{V}(K) \to A$, $K \in ob(\mathcal{K})$, such that for nay elementary morphism $f : K \to K'$ conditions described in Fig. 12 hold:

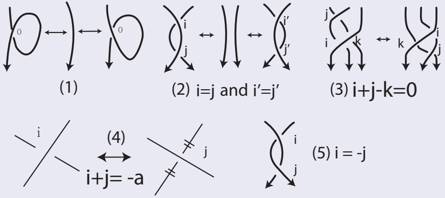

Figure 12:

Example 2.8

The label, which is defined in the previous section, is a winding parity with an abelian group $\mathbb{Z}$ (or $\mathbb{Z}_n$) and fixed element $a = 1$.

Example 2.9

The Gaussian parity is a winding parity with an abelian group $\mathbb{Z}_2$ and the fixed element $a = 0$.

Homological winding parity for crossings of $S_g \times S^1$

Let us define labels for crossings of $S_g \times S^1$ as follows:
Let c be a crossing of a diagram with double lines of a knot in $S_g \times S^1$. Let us consider its local pre-images in $S_g \times [0, 1]$ and we connect them by straight lines as described in Fig. 13. By connecting the straight line with the arc beginning from c to c, we obtain a half γ_c. Let us define the homological winding parity by $[\gamma_c] \in H_1(S_g \times S^1)/[K]$, where $[K]$ is the equivalence class of K in $H_1(S_g \times S^1)/[K]$.

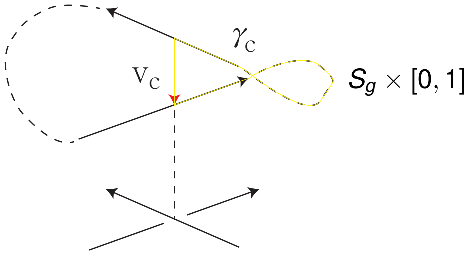

Figure 13: A half γ_c of a crossing c

Theorem 2.10

The homological winding parity is a winding parity with the abelian group $H_1(S_g \times S^1)/[K]$ with the fixed element $a = -[\{*\} \times S^1]$.

By the Künneth theorem, $H_1(S_g \times S^1) \cong H_1(S_g) \oplus H_1(S^1)$. Let $p_1 : H_1(S_g \times S^1) \to H_1(S_g)$ and $p_2 : H_1(S_g \times S^1) \to H_1(S^1)$ be projections. Then we obtain that

$$\begin{aligned} H_1(S_g \times S^1)/[K] &\to H_1(S_g \times S^1)/([p_1(K)] \oplus [p_2(K)]) \\ &\cong H_1(S_g)/[p_1(K)] \oplus H_1(S^1)/[p_2(K)] \end{aligned}$$

Corollary 2.11

$\{p_2([\gamma_c])\}$ *is the label described in Lemma 2.6.*

Corollary 2.12

$\{sgn(c)p_1([\gamma_c])\}$ *is an oriented parity.*

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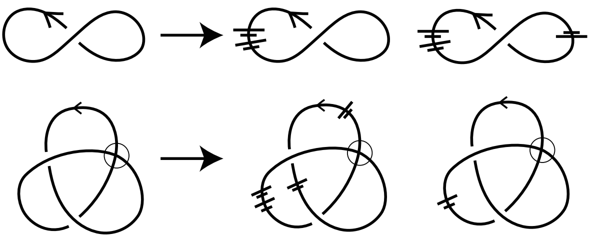
In this section we apply winding parities, in particular the label of crossings to classify knots in $S_g \times S^1$. We can obtain candidates of knots in $S_g \times S^1$ by putting double lines on virtual knot diagram. We call a diagram D^{dl} with double lines obtained from a virtual knot diagram D by putting double lines *a diagram with double lines based on D*, or simply, *a diagram based on D.* We will consider diagrams based on virtual knot diagrams with small number of crossings.



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We consider diagrams based on virtual knot diagrams with one crossing with a crossing sign $\epsilon = \pm 1$ (simply, we call them diagrams with one crossing). We present neighboring double lines by one line with an integer number *m*, where *m* is the sum of signs of neighboring double lines, as illustrated in Fig. 14.

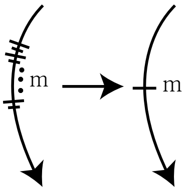

Figure 14: The neighboring double lines with sum of signs of neighboring double lines on an arc is presented by one line with the integer number *m*

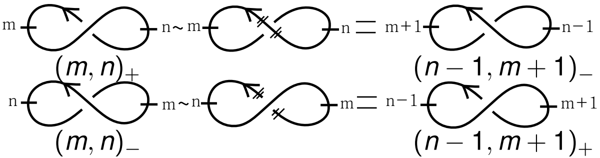

Figure 15: Knot diagrams with one crossing corresponding to $(m, n)_\epsilon$ and their equivalence relation

From the moves for diagrams with double lines, we obtain that $(m, n)_\epsilon$ is equivalent to $(n-1, m+1)_{-\epsilon}$. Note that the degree of $(m, n)_\epsilon$ is equal to $m + n$.

Lemma 3.1

If $(m, n)_\epsilon$ is equivalent to $(0, n)_\delta$ or $(m, 0)_\delta$ for $\epsilon, \delta = \pm 1$, then it is equivalent to the trivial diagram with a line with an integer number $m + n$.

Proof.

It is clear that $(0, n)_\delta$ and $(m, 0)_\delta$ are equivalent to the trivial diagram with double lines by move (1). By using our notation, we can replace double lines by a line with the integer number $m + n$. □

Theorem 3.2

Let m, n be integers such that $|m+n| \neq 1, 2$. If $(m, n)_\epsilon$ is not equivalent to $(0, n)_+$ and $(0, m+1)_-$ modulo $|m+n|$, then it is not equivalent to a trivial diagram.

Idea of proof. The label of the crossing of $(m, n)_+$ is m. By applying moves on $(m, n)_\epsilon$, the label of the crossing remains m or is changed to $-m-1 \equiv n+1$ modudo $|m+n|$. Since $m \not\equiv 0$ and $n+1 \not\equiv 0$ modulo $|m+n|$, it cannot be canceled by the first move.

Example 3.3 ($deg = 0$ case)

Let $m \neq 0, -1$ be an integer. Then $(m, -m)_+$ is not equivalent to $(0, -m)_+$ and $(0, m+1)$. From Theorem 3.2 it follows that $(m, -m)_+$ is not equivalent to the trivial diagram.

Example 3.4 ($deg \neq 0$ case)

Let k be integers with $|k| \geq 3$. For $1 \leq m \leq k-2$, $(m, k-m)_+$ is not equivalent to $(0, k-m)_+$ and $(0, m+1)_-$ modulo k. From Theorem 3.2 it follows that $(m, k-m)_+$ is not equivalent to the trivial diagram.

Corollary 3.5

For any $m, n \geq 1$ with $m \neq n$, $(m, -m)_+$ and $(n, -n)_+$ are not equivalent. Therefore, there are infinitely many nontrivial knots of degree 0 with one crossing.

Corollary 3.6

For an integer k with $k \geq 3$, there are at least

$$\begin{cases} \lfloor \frac{k-2}{2} \rfloor + 1 & \text{if } k \text{ is odd} \\ \lfloor \frac{k-2}{2} \rfloor & \text{if } k \text{ is even} \end{cases}$$

nontrivial knots of degree k with one crossing.

Remark 3.7

When $k = 1$ or $k = 2$, we cannot apply the same statement by using labels.

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We are interested in diagrams based on virtual knot diagrams with two crossings, or simply we call them diagrams with two crossings. There are two possible knot diagrams with two crossings described in Fig. 16.

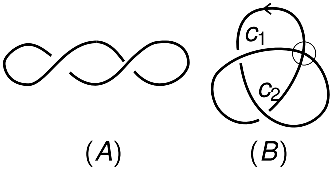

Figure 16: Virtual knot diagrams with two crossings: (A) A diagram with two kinks, (B) (Positive) virtual trefoil

A condition to be trivial

Let D be a diagram with double lines based on the diagram (B) in Fig. 16. Let a (and b) be the arc from undercrossing of c_1 (c_2, respectively) to overcrossing of c_2 (c_1, respectively). Let $deg(a)$ and $deg(b)$ be the sum of signs of double lines on a and b respectively. Assume that $deg(a) - deg(b) = -1$.

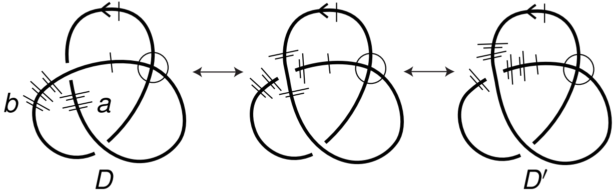

Figure 17: $deg(a) = 2$ and $deg(b) = 3$

Note that by move (4) and (b) one can obtain the move described in Fig. 18.

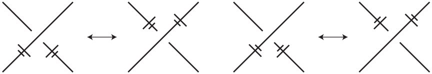

Figure 18: Two double lines pass through the crossing without crossing change

By applying this move one can obtain a diagram D' from D so that $deg(a) = 0$ and $deg(b) = 1$ in D' as described in Fig. 19. It is equivalent to the trivial diagram.

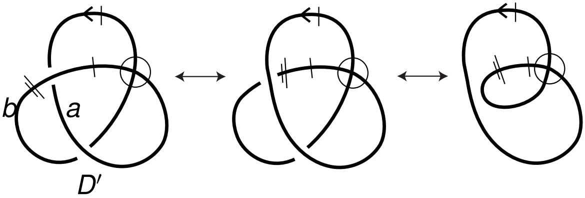

Figure 19:

Remark 3.8

For labels $p(c_1)$ and $p(c_2)$ of crossings c_1 and c_2, by definition of labels of crossings, it holds that
$p(c_1) + p(c_2) - deg(a) + deg(b) = deg(D) \equiv 0$ modulo $deg(D)$.
Therefore, the assumption in the previous page gives rise to the condition $p(c_1) + p(c_2) \equiv -1$ modulo $deg(D)$. In particular, if $deg(D) = 0$, then $deg(a) - deg(b) = -1$ if and only if $p(c_1) + p(c_2) = -1$.

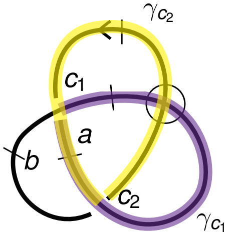

Figure 20: $\gamma_{c_1} * a^{-1} * \gamma_{c_2} * b \sim D$

Theorem 3.9

Let D be a diagram based on the diagram (B) in Fig. 16 and $k = deg(D)$. Assume that

1. $|k| \geq 3$,
2. $p(c_1) + p(c_2) \not\equiv -1$ *modulo* k,
3. *one of* $p(c_i)$, $i = 1, 2$, *is not equal to* 0 *modulo* k,
4. *one of* $p(c_i)$, $i = 1, 2$, *is not equal to* -1 *modulo* k.

Then D is not equivalent to the trivial diagram.

Corollary 3.10

There are infinitely many nontrivial knots of degree 0 *with one crossing.*

Corollary 3.11

For an integer k with $k \geq 3$, there are at least $\lfloor \frac{k-2}{2} \rfloor$ nontrivial knots of degree k based on the diagram (B) in Fig. 16.

Contents

Further research 1

- The labels of crossings of knots in $S_g \times S^1$ of degree 1 and 2 does not work well.

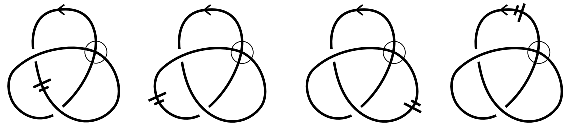

Figure 21: Diagrams with one double line

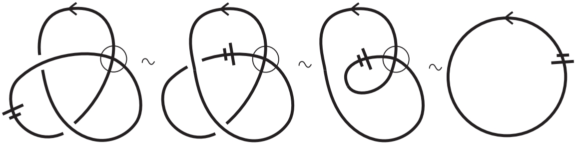

Figure 22: The second diagram on the above list is equivalent to trivial

Further research 2

• To construct invariants by using winding parity from invariants for virtual knots:

Let K be a diagram (a framed 4-graph). Then its parity bracket $[K]$ is defined by the formula

$$[K] = \sum_{s_{even,1}} K_{s_{even,1}}.$$

Theorem 4.1

The bracket $[\cdot]$ is an invariant of free knots.

The most important formula for this bracket is $[K] = K$ for diagrams with all chords odd. Since the graph is odd, its bracket $[K]$ consists of only one summand — graph K itself. This graph is its own minimal representative. It follows that if a free knot has only odd crossings, then it is non-trivial.

Further research 3

- To extend notions for knots in $S_g \times S^1$ to knots in $\Sigma \times S^1$ where Σ is an oriented surface with boundary.

Definition 4.1

A fibered link in S^3 is a collection of disjointly embedded circles $L = L_1 \sqcup \cdots \sqcup L_k$ such that $S^3 - N(L)$ is diffeomorphic to the quotient of $F \times I$ by an equivalence $(x, 0) \sim (h(x), 1)$ where h is a diffeomorphism of F with $h|\partial F = Id$, where F is a Seifert surface of L.

In [2] M. Chrisman and V.O. Manturov studied virtual knots by using 2-component link $K \sqcup J$ with $lk(K, J) = 0$, where J is a fibered knot:

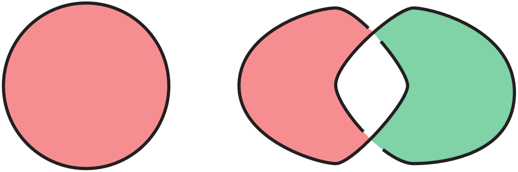

Figure 23: Seifert surfaces of a link

Thank you!

References I

- S. Carter, S. Kamada, M. Saito, *Stable Equivalence of Knots on Surfaces and Virtual Knot cobordisms,* Journal of Knot Theory and Its Ramifications, Vol. 11, No. 03, pp. 311-322 (2002).
- M. Chrisman, V.O. Manturov: *Fibered Knots and Virtual Knots,* J. Knot Theory Ramifications, Vol. 22, Issue 12 (2013), 23 pp.
- S. Kim: *The Groups G_n^k with additional structures,* Matematicheskie Zametki, Vol. 103, No. 4 (2018), pp. 549 – 567.
- N. Kamada and S. Kamada, *Abstract link diagrams and virtual knots,* J. Knot Theory Ramifications. Vol. 09, No. 01, pp. 93-106 (2000).
- L. H. Kauffman: *Virtual Knot Theory,* European J. Combin. 20 (1999), 663 –690.

References II

- M. K. Dabkowski, M. Mroczkowski: *KBSM of the product of a disk with two holes and S^1*, Topol. Appl., Vol. 156, Issue 10, 1 June 2009, pp. 1831–1849
- I.M. Nikonov: *Parity functors,* arXiv:2109.12230.
- D.P. Ilyutko, V.O. Manturov, I.M. Nikonov: *Virtual Knot Invariants Arising From Parities,* arXiv:1102.5081 [math.GT]
- V.O. Manturov, D.P. Ilyutko: Virtual knots:the state of the art, World Scientific Publishing, 2012.
- I.M. Nikonov: *Weak parities and functorial maps,* J. Math. Sci. 214:5 (2016) 699 –717.