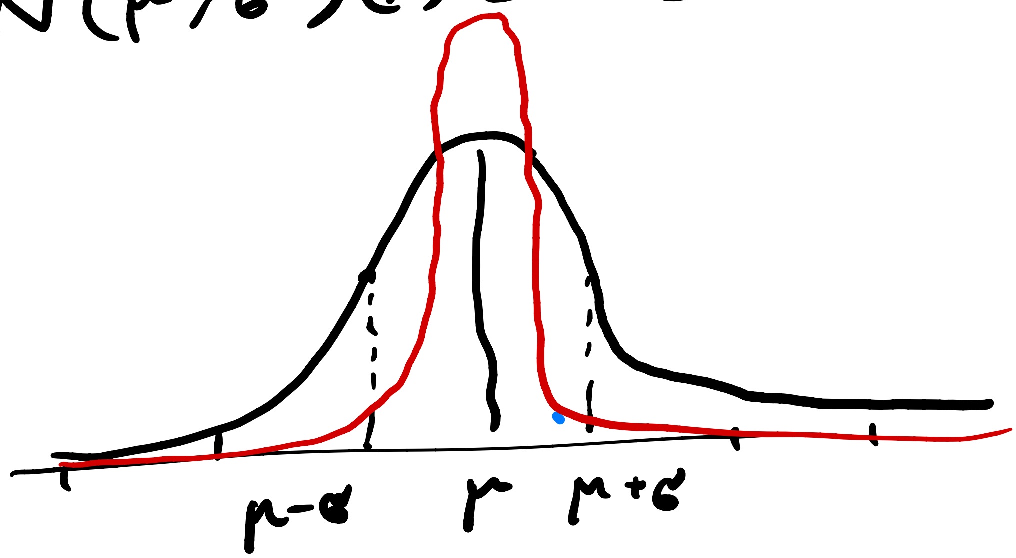
Integration over matrices

normal distribution

$$N(\mu, \sigma^2)(x) = C\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



you can talk about the normal dist over any vector space.

if you have a norm.

Our Normed vector space is Hermitian $N \times N$ matrices.

$$A = \begin{pmatrix} a_{11} & z_{12} & z_{13} & \cdots \\ & a_{22} & \cdots & \vdots \\ & & \ddots & \vdots \\ & & & a_{NN} \end{pmatrix}$$

$$A^t = A^c$$

Dim of Hermitian matrices.

$$N + 2\frac{N(N-1)}{2} = N + N^2 - N$$
$$= N^2$$

Hermitian matrices $\cong \mathbb{R}^{N^2}$

use the usual norm of $\mathbb{R}^{N^2}$?

No because this norm is not natural - it is not invariant under the action of the unitary group.

$$H \rightsquigarrow UHU^{-1}$$

$$|(a_{ii}, z_{ij})| = \sum_{i<j} \underbrace{(a_{ii})^2 + 2|z_{ij}|^2}_{\mathrm{tr}\,A^2}$$

$$U(N) \longrightarrow O(N^2)$$

$$\mathrm{tr}\,A^2 \stackrel{?}{=} \mathrm{tr}\,(UAU^{-1})^2$$

$$\| $$

$$\mathrm{tr}(UA\cancel{U^{-1}}\cancel{U}AU^{-1})$$

$$\|$$

$$\mathrm{tr}\,A^2$$

$\Rightarrow$ we have a norm on $\mathcal{h}$

$\shortparallel$

hermitian

$N \times N$ matrices.

given by $\operatorname{tr} A^2$

$$C e^{-\frac{\operatorname{tr} A^2}{2}}$$

A Normal distribution on $\mathcal{h}$

invariant under the action

of unitary group

$\Downarrow$

V: complex N-dim vector space.

with a Hermitian metric $\operatorname{herm}(V, V)$

has a natural normal dist

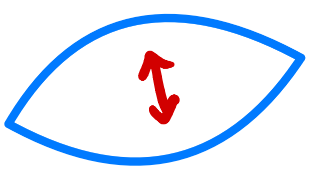
We are interested in expected values of

$$\frac{\displaystyle\int_{\mathfrak{h}} (\operatorname{tr} A^3)(\operatorname{tr} A^5)(\operatorname{tr} A^2)\, c\, e^{-\frac{\operatorname{tr} A^2}{2}}\, d\mathrm{Vol}}{\displaystyle\int_{\mathfrak{h}} \operatorname{tr} A^3\, c\, e^{-\frac{\operatorname{tr} A^2}{2}}\, d\mathrm{Vol}}$$

dVol: A translation invariant vol on $\mathfrak{h}$

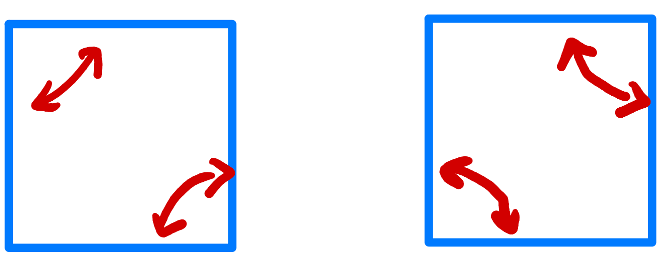
In how many diff ways you can glue the edges of a $2n$-gon to obtain a sphere?

$n = 1$ bi-gon



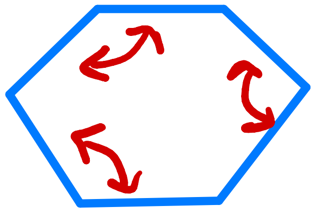
1 way

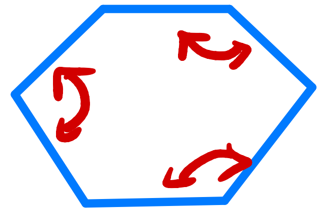
$n = 2$ square

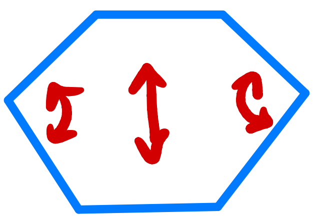


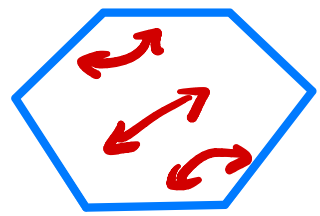
2 ways

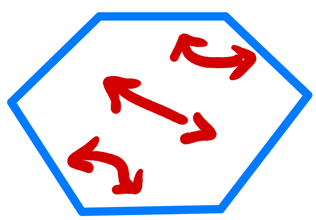
$n=3$ Hexagon











5 ways

$n=4$ Octagon

Try it! There are 14 ways.

HW Draw all 14 ways for the 8-gon!

Catalan numbers

HW Stasheff polytops. A_∞

(xy) 1

$(xy)z \quad x(yz)$ 2

$xyzw$ 5

$xyzwu$ 14

Catalan numbers.

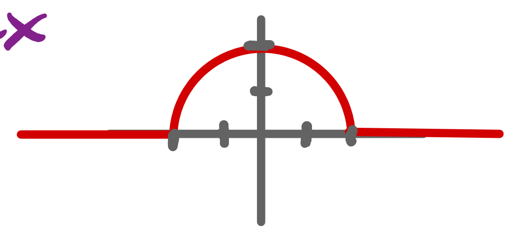
These numbers 1, 2, 5, 14, ...

are ubiquitous.

- number of ways to parenthesize n letters, or number of vertices of Stasheff's polytopes

- moments of the semi-circle law

$$\int_{-2}^{2} x^{2n} \frac{\sqrt{4-x^2}}{2\pi}\, dx$$



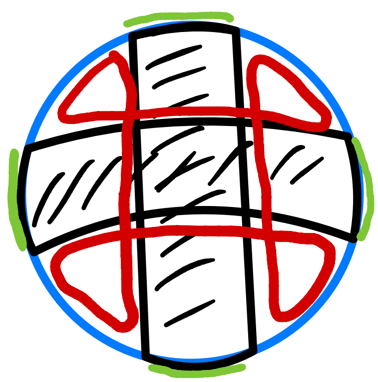
- $\dfrac{(2n)!}{n!(n+1)!}$ Catalan numbers

- # of ways to connect n ribbons to a disk with $2n$ disjoint intervals on it to get $n+1$ boundary components

Two Ribbons added to a disc so that we have three boundary circles

there are two ways

This doesn't count now since $\exists$ only one boundary.



$$1 - n = 2 - 2g - b$$

$$g = 0 \iff b = n+1$$

$$g = 1 \iff b = n-1$$

$$g = 2 \iff b = n-3$$

we can ask these questions about a 2n-gon for $g = 0, 1, 2, 3, \dots$

or alternatively for $b = n+1, n-1, n-3, \dots$

and organize this information in the following polynomials

n=0 0-gon $P_0(x) = 1x$

n=1 2-gon $P_1(x) = 1x^2$

n=2 4-gon $P_2(x) = 2x^3 + 1x$

n=3 6-gon $P_3(x) = 5x^4 + 10x^2$

n=4 8-gon $P_4(x) = 14x^5 + 70x^3 + 21x$

n=5 10-gon $P_5(x) = 42x^6 + 420x^4 + 483x^2$

$g=0$ or $b=n+1$ | $g=1$ or $b=n-1$ | $g=2$ or $b=n-3$

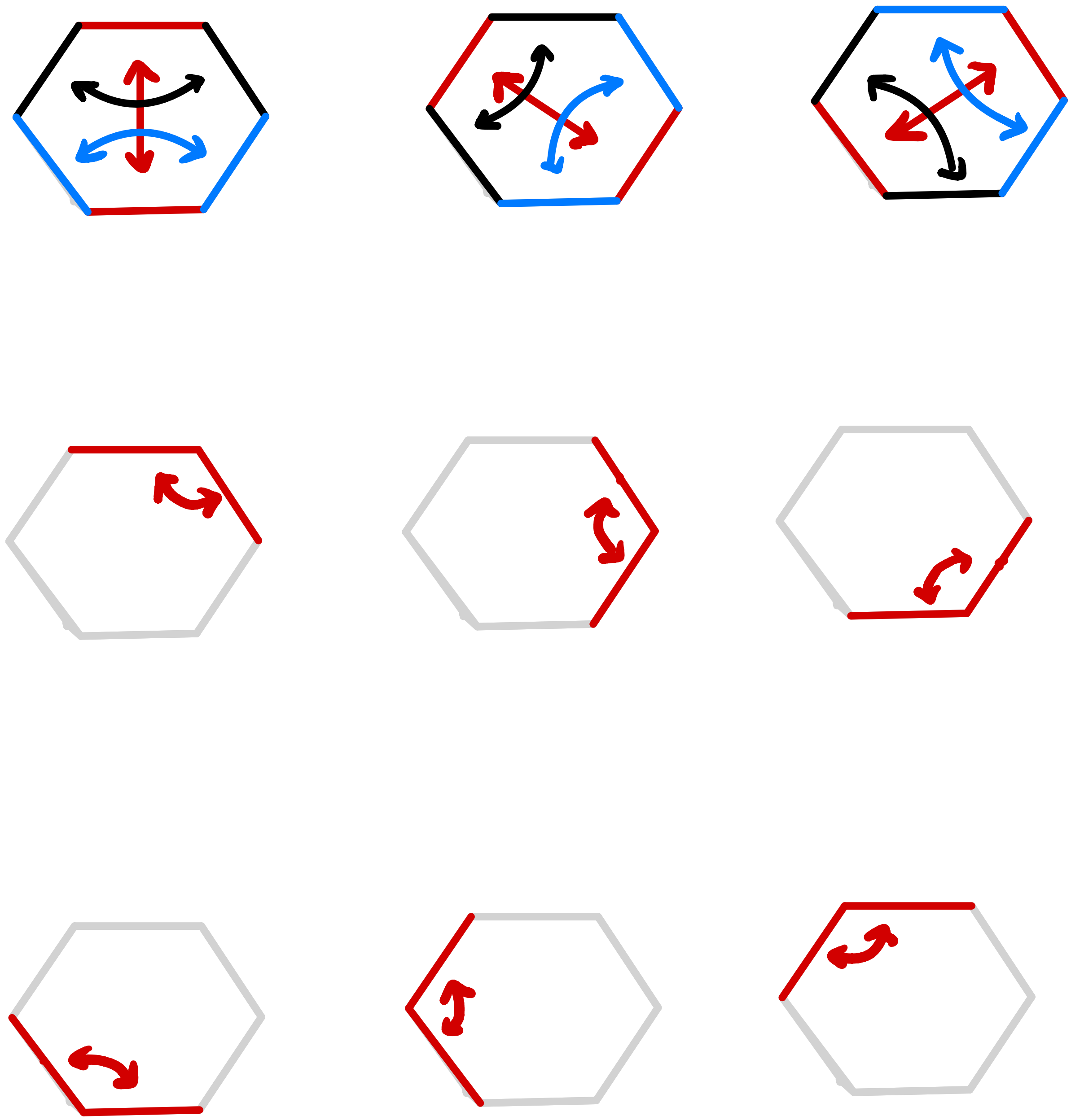
1×3 3

$1 \times 3 \times 5$ 15

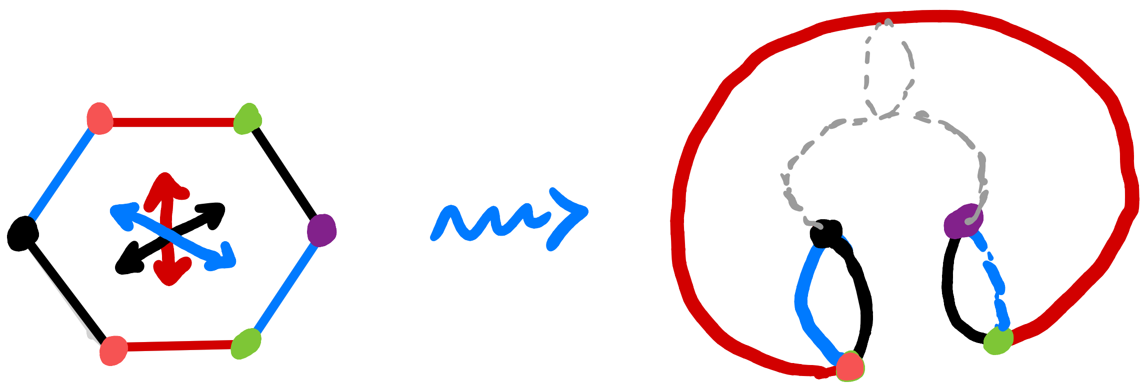
$1 \times 3 \times 5 \times 7$ 105

moments of the normal dist

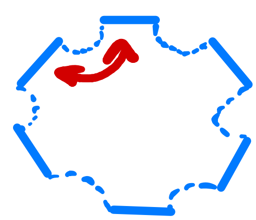
meaning of that "10"



There were 9 ways!



Rotate 180° & Glue!

you can also say there are 10 ways to connect 3 ribbons to  to obtain a surface with 2 boundary circles

$$1-\underset{\substack{\uparrow\\3}}{n} = 2-2\underset{\substack{\uparrow\\1}}{g}-\underset{\substack{\uparrow\\2}}{b}$$

These polynomials have a closed formula (Harer–Zagier) 1986 The Euler char of moduli space of curves.

$$P_n(x) = \frac{(2n)!}{2^n\, n!} \sum_{k=0}^{n} \binom{n}{k} 2^k \frac{x(x-1)\cdots(x-k)}{(k+1)!}$$

See Etingof's Note: Mathematical ideas and notions of QFT

They also satisfy and are determined by

$$\begin{cases} P_n(x) = \dfrac{4n-2}{n+1}\, x\, P_{n-1}(x) + \dfrac{(n-1)(2n-1)(2n-3)}{n+1} P_{n-2}(x) \\ P_0(x) = x\,, \quad P_1(x) = x^2 \end{cases}$$

(Harer–Zagier) 1986

Let's switch topics and talk a bit about Random Matrices (Gaussian Unitary Ensemble aka GUE)

$$h_N = N \times N \text{ Hermitian matrices}$$
$$= \{ A \mid \bar{A}^{tr} = A \}$$
$$\cong \mathbb{R}^{N^2}$$

as a vector space

$$N + 2\frac{N(N-1)}{2} = N^2$$

h_N has a norm which is not the Euclidean norm

$$|A| = \operatorname{tr} A^2 = \sum_i x_{ii}^2 + 2\sum_{i<j} x_{ij}^2$$

Has a natural Gaussian distribution

$$c\, e^{-\frac{\operatorname{tr} A^2}{2}}$$

with respect to which we can find the expected values of the traces of the powers of a matrix

$$\mathbb{E}(\operatorname{tr} A^{2n}) = \int_{\mathcal{H}_N} \operatorname{tr} A^{2n}\, c\, e^{-\frac{\operatorname{tr} A^2}{2}}\, dA$$

translation inv measure on the vector space $\mathcal{H}_N$

Thm Harer–Zagier 1986

$$\mathbb{E}(\operatorname{tr} A^{2n}) = P_n(N)$$

$$P_n(x) = \frac{(2n)!}{2^n\, n!} \sum_{k=0}^{n} \binom{n}{k} 2^k \frac{x(x-1)\cdots(x-k)}{(k+1)!}$$

$$\begin{cases} P_n(x) = \dfrac{4n-2}{n+1}\, x\, P_{n-1}(x) + \dfrac{(n-1)(2n-1)(2n-3)}{n+1} P_{n-2}(x) \\ P_0(x) = x\, ,\ P_1(x) = x^2 \end{cases}$$

(Harer–Zagier)
1986

n=0
0-gon $P_0(x) = 1x$

n=1
2-gon $P_1(x) = 1x^2$

n=2
4-gon $P_2(x) = 2x^3 + 1x$

n=3
6-gon $P_3(x) = 5x^4 + 10x^2$

n=4
8-gon $P_4(x) = 14x^5 + 70x^3 + 21x$

n=5
10-gon $P_5(x) = 42x^6 + 420x^4 + 483x^2$

$\uparrow$ g=0 or b=n+1

$\uparrow$ g=1 or b=n-1

$\uparrow$ g=2 or b=n-3

$$\int_{\mathcal{H}_N} \operatorname{tr} A^{2n} \, C e^{-\frac{\operatorname{tr} A^2}{2}} \, dA = P_n(N)$$

Note that when $N=1$ we have the usual moments of standard normal distribution

$$\int_{-\infty}^{+\infty} x^{2n} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = (2n-1)!!$$

$1x$

$1x^2$

$2x^3 + 1x$

$5x^4 + 10x^2$

$14x^5 + 70x^3 + 21x$

$42x^6 + 420x^4 + 483x^2$

$1 = 1!!$

$3 = 3!!$

$15 = 5!!$

$105 = 7!!$

$945 = 9!!$

Wigner's (Semi-Circle Law)

$$\lim_{N\to\infty} \underbrace{\frac{1}{N}\int \operatorname{tr} f\left(\frac{A}{\sqrt{N}}\right) c\, e^{-\frac{A^2}{2}}\, dA}_{h_N} = \int_{-\infty}^{+\infty} f(x)\, \frac{\sqrt{4-x^2}}{2\pi}\, dx$$

moments of the semi-circle law when $f(x) = x^{2n}$

Polyvector field & divergence operator

$$\int \underline{\underline{\mathrm{div}}}(X) = 0$$

M manifold of dim n

ω volume form.

$$\begin{tikzcd}
\Gamma(\Lambda^\bullet TM) \arrow[r, "\omega \lrcorner", "\text{isom}"'] \arrow[d, phantom, "\shortparallel"] & \Gamma(\Lambda^\bullet T^*M) \arrow[d, phantom, "\shortparallel"] \\
\mathfrak{X}_\bullet & \Omega^\bullet
\end{tikzcd}$$

$$\begin{tikzcd}
f \in \mathfrak{X}_0 \arrow[r, mapsto] & f\omega \in \Omega^n \\
X \in \mathfrak{X}_1 \arrow[u, "\mathrm{div}"] \arrow[r, mapsto] & i_X \omega \in \Omega \arrow[u, "d_{n-1}"'] \\
X \wedge Y \in \mathfrak{X}_2 \arrow[u, "\mathrm{div}"] \arrow[r, mapsto] & i_{X\wedge Y}\omega \in \Omega \arrow[u, "d_{n-2}"']
\end{tikzcd}$$

$$M = \mathfrak{h}_N$$

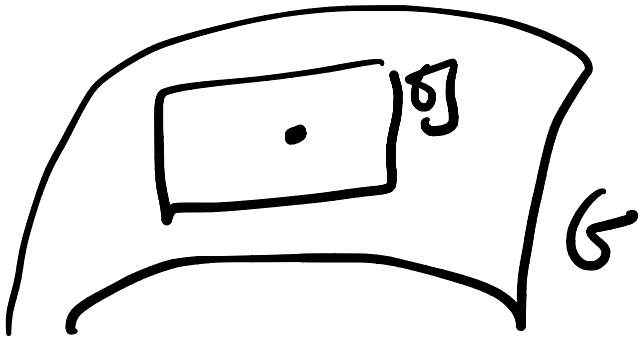
$$\omega = C\, e^{-\frac{\operatorname{tr} A^2}{2}} \mathrm{Vol}$$

translation inv volume.

$$\cdots \to \mathfrak{X}_2(\mathfrak{h}_N) \xrightarrow{\operatorname{div}} \mathfrak{X}_1(\mathfrak{h}_N) \xrightarrow{\operatorname{div}} \mathfrak{X}_0(\mathfrak{h}_N)$$

$$\mathfrak{X}_0(\mathfrak{h}_N) \cong \mathbb{R}^{N^2}$$

Chevalley Eilenberg of a differential graded Lie algebra



$$(\Lambda^\bullet \mathfrak{g}^*, d) \overset{\cong}{\hookrightarrow} (\Omega^\bullet G, d)$$

$$d = [\,,\,]^*$$

$$[\,,\,] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$$

$$\begin{tikzcd}
{[\,,\,]^* : \mathfrak{g}^*} \arrow[r] \arrow[d, phantom, "\cap" rotate=90] \arrow[dr, dashed] & \mathfrak{g}^* \wedge \mathfrak{g}^* \arrow[d, phantom, "\cap" rotate=90] \\
\Lambda^\bullet \mathfrak{g}^* \arrow[r, dashed, "d"'] & \Lambda^\bullet \mathfrak{g}^*
\end{tikzcd}$$

$d \leftarrow$ derivation

Proof

$d\omega(X, Y)$

$= \cdots$

$$\Lambda^\bullet \mathfrak{g}^* = \underline{\underline{S^\bullet(\mathfrak{g}[1])^* = CE(\mathfrak{g})}}$$

works for any $\mathfrak{g}$ that is graded and has a diff dgla

$$d : S^\bullet(\mathfrak{g}[1])^* \circlearrowleft$$

$$d = d_{\mathfrak{g}}^* + [\ ,\ ]^*$$

Claim

$$\mathcal{X}_{\bullet}(h_N) \cong S^{\bullet}(\mathfrak{gl}_N(A)[1])^*$$

$A = \{a, b\}$

$|a| = 1$

$|b| = 2$

$a^2 = 0$

$b^2 = 0$

$ab = 0$

$d(a) = b$

$d(b) = 0$

In this talk I want to explain this using cyclic cohomology and the Loday-Quillen-Tsygan map.

Let A be an A_∞-algebra

LQT gives a map

$$\mathrm{Sym}(\mathrm{Cyc}^\bullet(A)) \longrightarrow CE^\bullet(\mathfrak{gl}_N(A))$$

when $N \to \infty$ we get quasi-isom of "homotopy" Hopf-algebras map ?

$$\mathrm{Cyc}^\bullet(A) \sim \mathrm{Cyc}^\bullet(M_{N\times N}(A))$$

$$\mathrm{Cyc}^\bullet(M_{N\times N}(A)) \longrightarrow CE^\bullet(\mathfrak{gl}_N(A))$$

C_n-coinvariants of T(N×N matrices on A)

S_n-coinvariants of T(N×N matrices on A)

Thm when A = cyclic A_∞-alg of dim 3, then LQT map is a map of BV algebras.

What is the BV str on both sides?

One Sym $\mathrm{Cyc}^\circ(A)$

$\mathrm{Cyc}(A)$ is a Lie alg (String Top Bracket (Moshev, Kontsevich, Tradler Z., Kaufmann) Goldman)

working with the augmented cyclic cohomology

$\Longrightarrow$ Sym $\mathrm{Cyc}^\circ(A)$ is a Lie alg

Chevalley Eilenberg chain

$\mathrm{Cyc}^\circ(A)$ is a Lie coalgebra (String Top Cobracket Tradler Z. Kaufmann Turaev)

$\Longrightarrow$ derivation on Sym $\mathrm{Cyc}^\circ(A)$

- Lie bracket (deg +2)
- Lie cobracket (deg −2)
- compatibility (deg 0) Drinfeld + involutive

$\Longleftrightarrow$ $(\partial + \Delta)^2 = 0$

BV str on the right side.

M manifold w/ volume element ω

Koszul 1985

$$\mathfrak{X}_\bullet(M) \xrightarrow{\sim} \Omega^{n-\bullet}(M)$$

$$X \longmapsto i_X \omega$$

transport of d = divergence

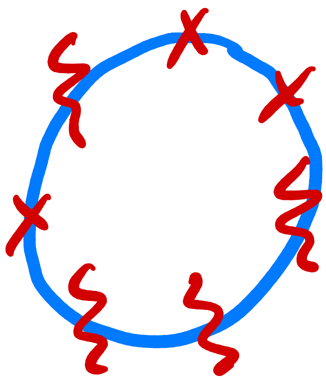
$$\cdots \longrightarrow \mathfrak{X}_2(M) \xrightarrow{\text{div}} \mathfrak{X}_1(M) \xrightarrow{\text{div}} \mathfrak{X}_0(M) \to 0$$

If you like Hodge theory

$$\cdots \xrightarrow{d^*} \Omega^2(M) \xrightarrow{d^*} \Omega^1(M) \xrightarrow{d^*} \Omega^0(M) \to 0$$

$$A = \mathbb{C}[a,b] \big/ \langle a^2, b^2, ab \rangle$$

$$\langle a, b \rangle = 1$$



$$\begin{array}{rcc} & a & b \\ & \updownarrow & \updownarrow \\ A^*[-1] = & x, & \xi \\ & \uparrow & \uparrow \\ & \deg 0 & \deg -1 \end{array}$$

$$d(b) = -a$$

when $A = \mathbb{C}[a,b]/(a^2, b^2, ab)$ $\qquad |a| = 1$, $|b| = 2$

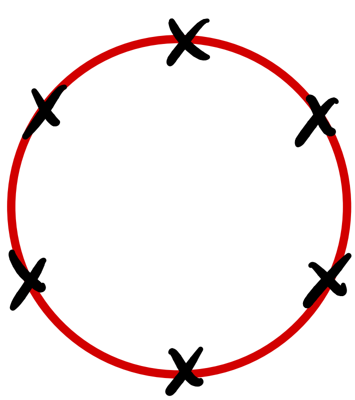
$da = b$

$\langle a, b \rangle = 1$

$$\underline{CE^\bullet(\mathfrak{gl}_N(A))} \underset{BV}{\cong} \mathcal{X}_\bullet(\underbrace{h_N}) \underset{\mathbb{R}}{\otimes} \mathbb{C}$$

w/r/t Gaussian divergence

Punchline:



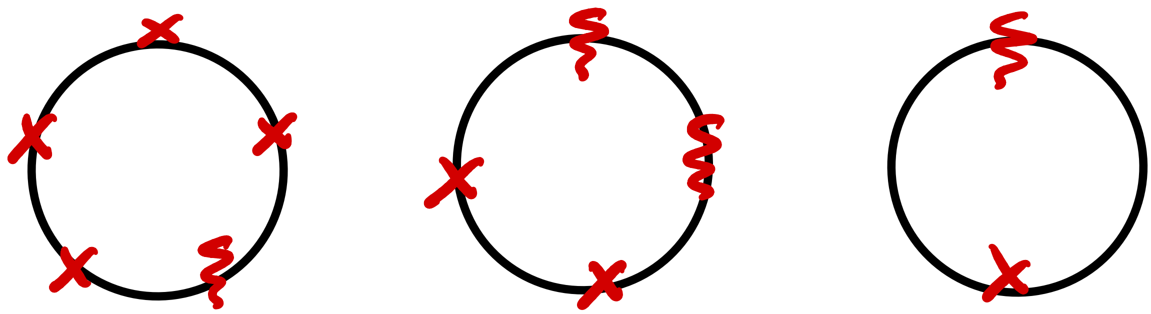
$$\longmapsto \quad \mathrm{tr}\, A^6$$

$x \longleftrightarrow a$ $\qquad d\xi = -x$

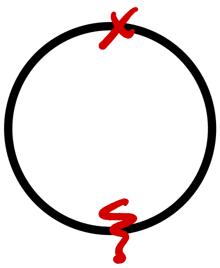
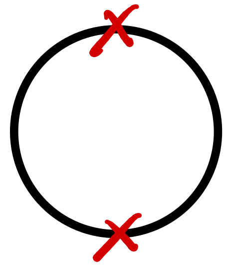
$|x| = 0 \quad |a| = 1$ $\qquad \langle x, \xi \rangle = 1$

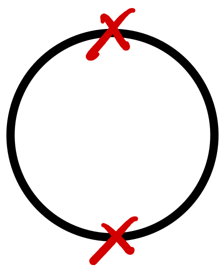
$|\xi| = 1 \quad |b| = 2$

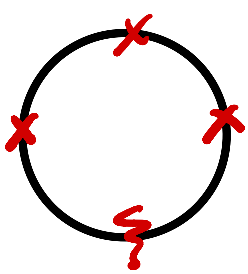
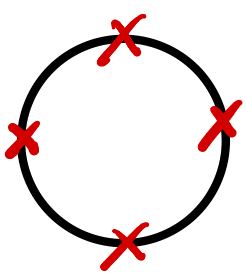
What are the rules of D on $\mathrm{Sym}(\mathrm{Cyc}^{\bullet} A)$?

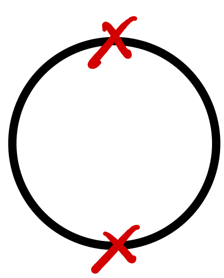
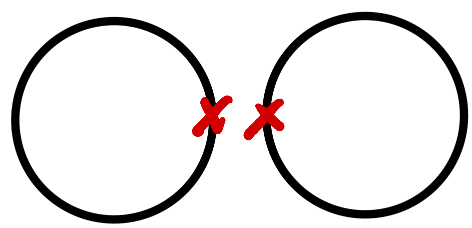


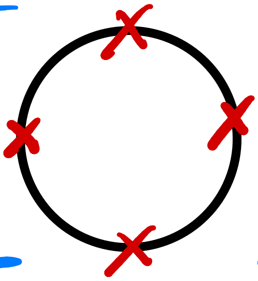
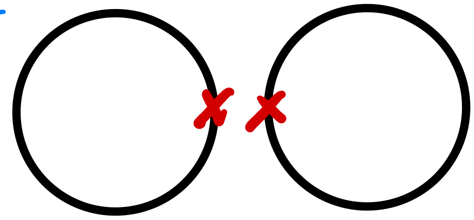
- empty circles $= N$
- change a ξ to $-x$
- Let a ξ eat an x and read off what's left.
  - Two circles $\rightsquigarrow$ one circle
    String Bracket (Chas Sullivan, Goldman)
  - one circle $\rightsquigarrow$ two circles
    String Cobracket (Chas Sullivan, Turaev)

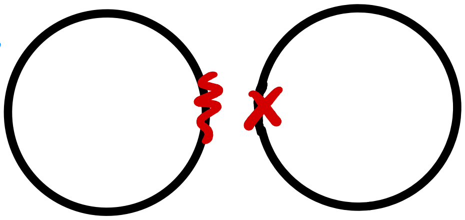
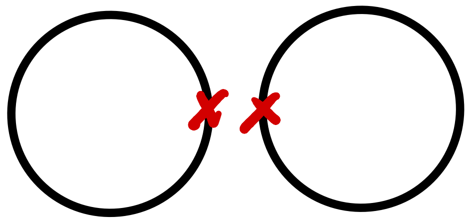
$D\Big($  $\Big) = -$  $+ N^2$

$\Big[$  $\Big] = N^2 = P_1(N)$

$D\Big($  $\Big) = -$ 

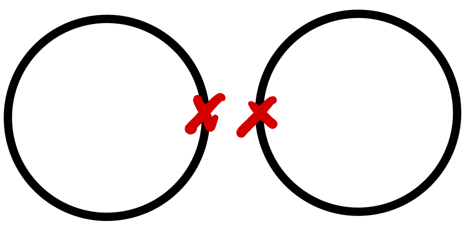
$+ 2N$  $+$ 

[] $= 2NN^2 +$ []

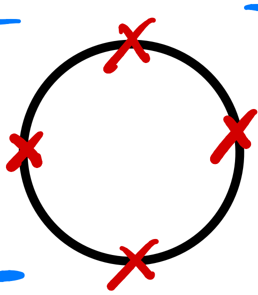
$D($  $) = -$ 

$+ N$

Therefore

[] $= N$

therefore

[] $= 2NN^2 + N$

$= 2N^3 + N = P_2(N)$

$$P_0(x) = 1x$$

$$\boxed{P_1(x) = 1x^2}$$

$$\boxed{P_2(x) = 2x^3 + 1x}$$

$$P_3(x) = 5x^4 + 10x^2$$

$$P_4(x) = 14x^5 + 70x^3 + 21x$$

$$P_5(x) = 42x^6 + 420x^4 + 483x^2$$

there is a similar story for the orthogonal ensemble where

cyclic coh $\longleftrightarrow$ dihedral coh

$$CE(gl_N) \longleftrightarrow CE(O_N)$$

Multi-Trace Calculations

Fun elementary cyclic calc calculation.

$$\langle (\mathrm{tr}A)^{2n} \rangle = N^n (2n-1)!!$$

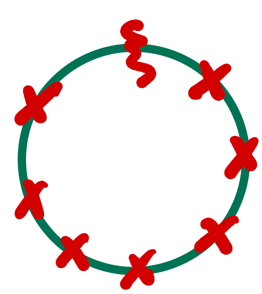
$$\langle (\mathrm{tr}A)^{2n}\, \mathrm{tr}(A^2) \rangle = N^2 (2n+N^2)(2n-1)!!$$

$$\langle (\mathrm{tr}A^2)^{n+1} \rangle = \begin{cases} (2n+1)!! & N=1 \\ \dfrac{(N^2+2n)!!}{(N^2-2)!!} & N \text{ odd} > 1 \\ \dfrac{2^{n+1}\left(\frac{N^2}{2}+n\right)!}{\left(\frac{N^2}{2}-1\right)!} & N \text{ even} \end{cases}$$

Proposition

$$\sum_{k=1}^{2n-1} \langle \mathrm{tr}A^k\, \mathrm{tr}A^{2n-k} \rangle = \langle \mathrm{tr}A^{2n+2} \rangle + 2N \langle \mathrm{tr}A^{2n} \rangle$$

Proof: $n=3$,



More interestingly.

$$\lim_{N\to\infty}\left[\frac{\int_{\mathfrak{h}_N}\mathrm{Tr}(X^{10})\mathrm{Tr}(X^{42})\mathrm{Tr}(X^{15})\mathrm{Tr}(X^{43})\mathrm{Tr}(X^{47})\mathrm{Tr}(X^{63})e^{-\frac{1}{2}\mathrm{Tr}(X^2)}\mathrm{d}X}{N^{112}\int_{\mathfrak{h}_N}e^{-\frac{1}{2}\mathrm{Tr}(X^2)}\mathrm{d}X}\right]=C_{5,21}A_{7,21,23,31}$$
$$=25081904924688737847061935982290890890757044619026344345600000.$$

$C_{k_1 \dots k_n}$ & $A_{l_1 \dots l_m}$ can be expressed explicitly ('C's more explicitly than 'A's)

Generalization of Wigner's semi-circle law

$$\lim_{N\to\infty}\left(\frac{\int_{\mathfrak{h}_N}\prod_{i=1}^{m}\left[\frac{1}{N}\mathrm{Tr}\left(q_i\left(\frac{X}{\sqrt{N}}\right)\right)\right]e^{-\frac{1}{2}\mathrm{Tr}(X^2)}\mathrm{d}X}{\int_{\mathfrak{h}_N}e^{-\frac{1}{2}\mathrm{Tr}(X^2)}\mathrm{d}X}\right)=\prod_{i=1}^{m}\left[\frac{1}{2\pi}\int_{-2}^{2}q_i(x)\sqrt{4-x^2}\mathrm{d}x\right].$$

For references see joint work with Owen Gwilliam, Gregory Ginot and Alastair Hamilton