

Intro to Model Categories

Motivation

Model categories provide an abstract framework in which to do homotopy theory. Oddly enough though, the starting point is not the concept of a "homotopy" but rather the more primitive notion of a "weak equivalence".

Homotopical Categories

Let \mathcal{C} be a category with a distinguished class of morphisms W which we call weak equivalences. We wish to think of these as giving a kind of weakened, unidirectional notion of isomorphism. In particular, we want to study constructions on and properties of \mathcal{C} that are invariant under weak equivalence.

Specifically, we want to study functors $\mathcal{C} \rightarrow \mathcal{D}$ taking elements of \mathcal{W} to isomorphisms of \mathcal{D} .

Example: The homology functor

$$H: \text{Top} \rightarrow \text{GrAb}$$

with $\mathcal{W} \subseteq \text{Mor}(\text{Top})$ the topological weak equivalences, i.e., maps inducing isomorphisms on homotopy groups. This functor takes weak equivalences to isomorphisms of homology groups.

It is natural to ask whether there is a universal such functor, i.e., one through which any other such functor factors uniquely:

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}[\mathcal{W}^{-1}] \\ & \searrow & \downarrow \\ & & \mathcal{D} \end{array} \quad \begin{array}{l} \text{think} \\ \text{localization} \\ \downarrow \end{array}$$

And indeed there is: $\mathcal{C}[\mathcal{W}^{-1}]$, also called the homotopy category

of $(\mathcal{C}, \mathcal{W})$ can be constructed as roughly follows:

$$\text{Let } \text{Ob}(\mathcal{C}[\mathcal{W}^{-1}]) = \text{Ob}(\mathcal{C})$$

Let $\text{Mor}(\mathcal{C}[\mathcal{W}^{-1}])$ consist of strings of composable arrows in $\text{Mor}(\mathcal{C})$ together with "formal inverses" of arrows in \mathcal{W} . Let composition be concatenation. Now impose the

$$\text{relations: } 1_a \sim (1_a)$$

$$(f, g) \sim (g \circ f)$$

$$(w, w^{-1}) \sim \downarrow_{\text{down}}$$

$$(w^{-1}, w) \sim \downarrow_{\text{up}}$$

(This is called localization - note the analogy with localization of rings).

The problem with $\mathcal{C}[\mathcal{W}^{-1}]$ is that it is usually huge and very difficult to work with. The solution given by model categories requires more structure than just \mathcal{W} ...

Preliminary Facts/Definitions

Lifting Problems

Let $i: A \rightarrow B$ and $p: X \rightarrow Y$ be morphisms in \mathcal{C} . A lifting problem is a commutative square:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ i \downarrow & & \downarrow p \\ B & \xrightarrow{g} & Y \end{array}$$

for any f and g . A solution to such a problem consists of a lift

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ i \downarrow & \nearrow \ell & \downarrow p \\ B & \xrightarrow{g} & Y \end{array}$$

$\ell: B \rightarrow X$ making the diagram commute.

If all such lifting problems have a solution, we say i left lifts against p and p right lifts against i , written lfp.

If $K \subseteq \text{Mor}(\mathcal{C})$ is a collection of morphisms, we denote:

$$K = \{ f \in \text{Mor}(\mathcal{C}) \mid f \circ k \forall k \in K \}$$

and

$$K^\circ = \{ g \in \text{Mor}(\mathcal{C}) \mid k \circ g \forall k \in K \}$$

Lastly, we write:

$$L \circ R$$

if

$$L = R^\circ \text{ and } R = L^\circ.$$

Def We say $f \in \text{Mor}(\mathcal{C})$ is a retract of $g \in \text{Mor}(\mathcal{C})$ if

f is a retract of g in the arrow category of \mathcal{C} , that is, if it fits into a commutative diagram:

$$\begin{array}{ccccc}
 & & \Delta_A & & \\
 & & \curvearrowright & & \\
 A & \rightarrow & C & \rightarrow & A \\
 f \downarrow & & \downarrow g & & \downarrow f \\
 B & \rightarrow & D & \rightarrow & B \\
 & & \Delta_B & & \\
 & & \curvearrowleft & &
 \end{array}$$

Closure properties of lifting classes

Classes of maps defined by lifting properties enjoy certain closure properties: Given $K \subseteq \text{Mor}(C)$,

$\square K$ is closed under $\begin{cases} \text{composition,} \\ \text{retracts,} \\ \text{pushouts} \end{cases}$

K^\square is closed under $\begin{cases} \text{compositions,} \\ \text{retracts,} \\ \text{pullbacks.} \end{cases}$

Proof for $\square K$:

Compositions: Let $f \in K$ and $g \in K$.

want a lift for

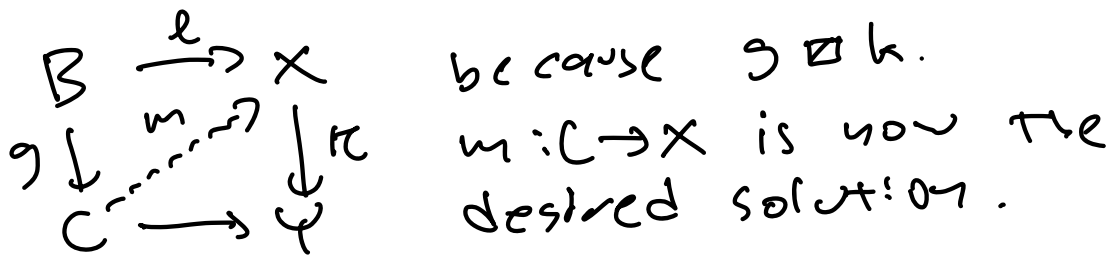
$$\begin{array}{ccc} A & \xrightarrow{\quad} & X \\ f \downarrow & & \downarrow k \\ B & & \\ g \downarrow & & \\ C & \xrightarrow{\quad} & Y \end{array}$$

for all $k \in K$.

First, we have

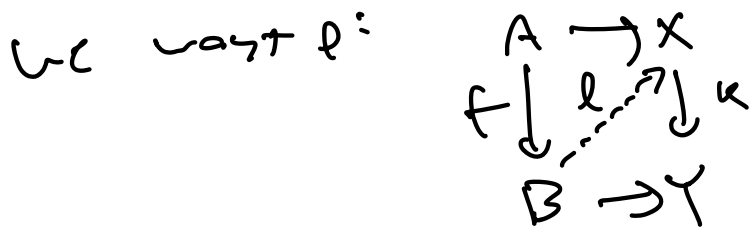
$$\begin{array}{ccccc} A & \xrightarrow{\quad} & X & & \\ f \downarrow & \ell \dashrightarrow & \downarrow k & & \\ B & \xrightarrow{\quad} & C & \xrightarrow{\quad} & Y \end{array}$$

because $f \in K$. Then we have

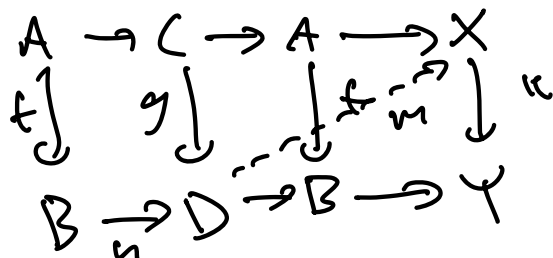


Retract

Say f is a retract of g , and $g \circ k$.



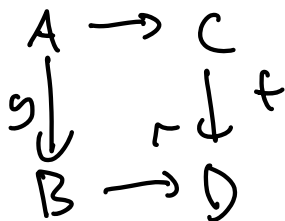
We have $m: D \rightarrow X$ because $g \circ k$:



But now $m \circ h: B \rightarrow X$ is the desired lift.

Pushouts

Say f is a pushout of g :



And $g \circ k$. Then we have $m: B \rightarrow X$:

$$\begin{array}{ccccc} A & \rightarrow & C & \xrightarrow{h} & X \\ g \downarrow & & \downarrow m & \dashrightarrow & \downarrow k \\ B & \rightarrow & D & \rightarrow & Y \end{array}$$

But by the universal property of a pushout, to the pair of maps $m: B \rightarrow X$ and $h: C \rightarrow X$ there corresponds (a unique) map $l: D \rightarrow X$. That is our lift. ■

The statements for k° follow similarly (or by duality).

Weak Factorization Systems

Def A weak factorization system on \mathcal{C} consists of a pair (L, R) with $L, R \subseteq \text{Mor}(\mathcal{C})$ satisfying:

1) (factorization) Any map $f \in \text{Mor}(\mathcal{C})$

factors as $A \xrightarrow{f} B$
 $\ell \in L, r \in R$, with $\ell \in L, r \in R$
 and

2) (lifting) $L \perp R$.

Model Category Axioms

Finally we can give the main definition:

Def. Let \mathcal{C} be a category with all small limits and colimits.

A model structure on \mathcal{C} consists of three classes of morphisms (C, F, W)

$\left(\begin{array}{l} C: \text{"cofibrations"}, \text{ denoted } \twoheadrightarrow \\ F: \text{"fibrations"}, \text{ denoted } \longrightarrow \\ W: \text{"weak equivalences"}, \text{ denoted } \xrightarrow{\sim} \end{array} \right)$

satisfying the axioms:

1) (2 out of 3) for any diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ & \searrow & \nearrow \\ & C & \end{array}$$
 two of the arrows are in W
 \Rightarrow the other is in W .

(as is the case for isomorphisms!)

2) (weak factorization systems)

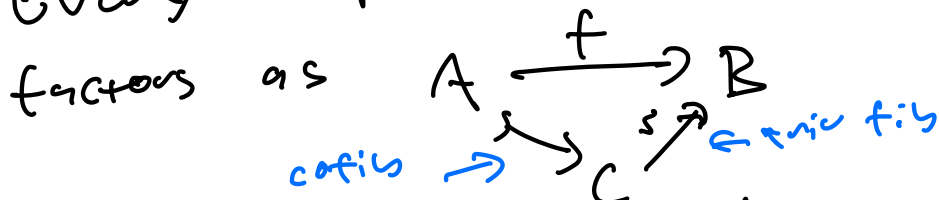
$(C, F \cap W)$ and $(C \cap W, F)$
form weak factorization systems.

Note: elements of $C \cap V$ are called trivial cofibrations ($\xrightarrow{\sim}$) and elements of $F \cap W$ are called trivial fibrations ($\xrightarrow{\sim}$),

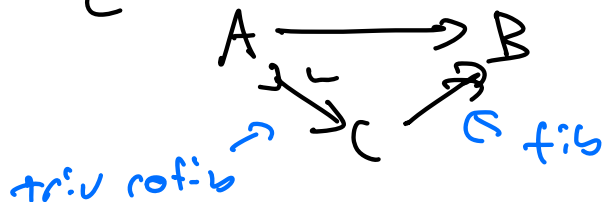
Quick consequences

- C (cofibrations) and $C \cap V$ (trivial cofibrations) are closed under compositions, retracts, and pushouts.
- F (fibrations) and $F \cap W$ (trivial fibrations) are closed under compositions, retracts, and pullbacks.

• Every map $f: A \rightarrow B \in \text{Mor}(C)$



and also as:



Cofibrant / Fibrant replacement

These are special "well-behaved" classes of objects:

Def An object X is cofibrant if the map $\emptyset \rightarrow X$ from the initial object of \mathcal{C} is a cofibration.

An object Y is fibrant if the map $Y \rightarrow *$ to the terminal object is a fibration.

Thm Every object X has a cofibrant replacement, i.e. a

weak equivalence

$$X^{cof} \xrightarrow{\sim} X$$

from a cofibrant object X^{cof} ,

and every object Y has a

fibrant replacement, i.e. a

weak equivalence

$$Y \xrightarrow{\sim} Y^{fib}$$

Proof Factor $\emptyset \rightarrow X$, and

Factor $Y \rightarrow *$.

Example 2

The category Top of compactly generated Hausdorff spaces has the Serre model structure in which

$$C = \left\{ \begin{array}{l} \text{(retracts of)} \\ \text{relative CW-complexes} \end{array} \right\}$$

$$F = \{ \text{Serre Fibrations} \}$$

$$W = \left\{ \begin{array}{l} \text{weak equivalences} \\ \text{(retracts of)} \end{array} \right\}$$

$$\text{Cofibrant objects} = \{ \text{CW-complexes} \}$$

$$\text{Fibrant objects} = \{ \text{everything} \}$$

The category $\text{Ch}_{\geq 0}$ of chain complexes has the projective model structure

$$C = \{ \text{injective chain maps with projective cokernel} \}$$

$$F = \{ \text{surjective chain maps in positive degree} \}$$

$$W = \{ \text{quasi-isomorphisms} \}$$

$$\text{Cofibrant objects} = \{ \text{complexes of projectives} \}$$

$$\text{Fibrant objects} = \{ \text{everything} \}$$

What about homotopies?

Def. For X an object in a model category, a cylinder object $Cyl(X)$ is an object fitting into the factorization

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{\text{fold}} & X \\ & \searrow & \nearrow \\ & & Cyl(X) \end{array}$$

of the fold map $X \sqcup X \rightarrow X$.

A path object $Path(X)$ is an object fitting into the factorization

$$\begin{array}{ccc} X & \xrightarrow{\text{diag}} & X \times X \\ & \searrow & \nearrow \\ & & Path(X) \end{array}$$

of the diagonal map $X \rightarrow X \times X$,

(In TOP the cylinder object is the cylinder and the path object is the path space).

Def. Given two maps $f, g: X \rightarrow Y$ we say f is left homotopic to g

$(f \sim_r g)$ if the map $(f, g): X \sqcup X \rightarrow Y$ extends to a map $h: C_1(X) \rightarrow Y$:

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{(f, g)} & Y \\ & \searrow & \uparrow \\ & & C_1(X) \end{array}$$

We say f, g are right homotopic $(f \sim_r g)$ if the map $(f, g): X \rightarrow Y \times Y$ factors through $\text{Path}(Y)$:

$$\begin{array}{ccc} X & \xrightarrow{(f, g)} & Y \times Y \\ & \searrow & \uparrow \\ & & \text{Path}(Y) \end{array}$$

Theorem If X is cofibrant and Y is fibrant, then left and right homotopy define equivalence relations on $\text{Hom}(X, Y)$ that coincide, and equivalence is preserved by composition, and $\text{Hom}(X, Y)_{/m}$ is preserved by weak equivalence.

This gives us a way of setting a handle on the homotopy category:

Theorem: Let \mathcal{C}^{cf} be the full subcategory of \mathcal{C} on the cofibrant-fibrant objects.

Then $\mathcal{C}^{cf}/\sim \cong \mathcal{C}[\omega^{-1}] (= Ho \mathcal{C})$
ie on these special objects we can get the homotopy category just by quotienting by the homotopy equivalence relation.

Finally, we can obtain the universal functor $\mathcal{C} \rightarrow Ho \mathcal{C}$ by precomposing with fibrant and cofibrant replacement functors so

$$Hom_{\mathcal{C}}(QRX, QRY) / \sim \cong Hom_{Ho(\mathcal{C})}(X, Y).$$