Intro to Model Categories

Motivation

Model categories provide an abstract framework in which to do homotopy theory. Oddly enough though, the starting point is <u>not</u> the concept of a "homotopy" but rather the more primitive notion of a "weak equivalence".

Homotopical Categories

Let $\mathcal{C}$ be a category with a distinguished class of morphisms W which we call weak equivalences. We wish to think of these as giving a kind of weakened, unidirectional notion of isomorphism. In particular, we want to study constructions on and properties of $\mathcal{C}$ that are invariant under weak equivalence.

Specifically, we want to study functors $\mathcal{C} \to \mathcal{D}$ taking elements of W to isomorphisms of $\mathcal{D}$.

<u>Example</u>: The homology functor

$$H : \mathrm{Top} \to \mathrm{GrAb}$$

with $W \subseteq \mathrm{Mor}(\mathrm{Top})$ the topological weak equivalences, i.e., maps inducing isomorphisms on homotopy groups. This functor takes weak equivalences to isomorphisms of homology groups.

It is natural to ask whether there is a universal such functor, i.e., one through which any other such functor factors uniquely:

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr] & \mathcal{C}[W^{-1}] \arrow[d, dotted] \\ & \mathcal{D} \end{tikzcd}$$

think localization

And indeed there is: $\mathcal{C}[W^{-1}]$, also called the <u>homotopy category</u>

of $(\mathcal{C}, W)$ can be constructed as roughly follows:

Let $Ob(\mathcal{C}[W^{-1}]) = Ob(\mathcal{C})$

Let $Mor(\mathcal{C}[W^{-1}])$ consist of strings of composable arrows in $Mor(\mathcal{C})$ together with "formal inverses" of arrows in W. Let composition be concatenation. Now impose the relations:

$$1_a \sim (1_a)$$
$$(f, g) \sim (g \circ f)$$
$$(w, w^{-1}) \sim 1_{\mathrm{dom}\, w}$$
$$(w^{-1}, w) \sim 1_{\mathrm{cod}\, w}$$

(This is called <u>localization</u> - note the analogy with localization of rings).

The problem with $\mathcal{C}[W^{-1}]$ is that it is usually <u>huge</u> and very difficult to work with. The solution given by model categories requires more structure than just W...

Preliminary Facts/Definitions

Lifting problems

Let $i: A \to B$ and $p: X \to Y$ be morphisms in $\mathcal{C}$. A <u>lifting problem</u> is a commutative square:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] & Y \end{tikzcd}$$

for any f and g. A solution to such a problem consists of a lift

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dotted, "\ell"] & Y \end{tikzcd}$$

$\ell: B \to X$ making the diagram commute.

If all such lifting problems have a solution, we say i left lifts against p and p right lifts against i, written <u>$i \boxslash p$</u>.

If $K \subseteq \mathrm{Mor}(\mathcal{C})$ is a collection of morphisms, we denote:

$$ {}^{\boxslash}K = \{ f \in \mathrm{Mor}(\mathcal{C}) \mid f \boxslash k \ \forall k \in K \}, $$

and

$$ K^{\boxslash} = \{ g \in \mathrm{Mor}(\mathcal{C}) \mid k \boxslash g \ \forall k \in K \}. $$

Lastly, we write:

$$ L \boxslash R $$

if

$$ L = {}^{\boxslash}R \ \underline{\text{and}} \ R = L^{\boxslash}. $$

<u>Def</u> We say $f \in \mathrm{Mor}(\mathcal{C})$ is a <u>retract</u> of $g \in \mathrm{Mor}(\mathcal{C})$ if f is a retract of g in the arrow category of $\mathcal{C}$, that is, if it fits into a commutative diagram:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "1_A"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r] \arrow[rr, bend right, "1_B"'] & D \arrow[r] & B
\end{tikzcd}
$$

Closure properties of lifting classes

Classes of maps defined by lifting properties enjoy certain closure properties: Given $K \subseteq \mathrm{Mor}(\mathcal{C})$,

$^{\boxslash}K$ is closed under $\left\{\begin{array}{l}\text{composition,} \\ \text{retracts,} \\ \text{pushouts}\end{array}\right.$

$K^{\boxslash}$ is closed under $\left\{\begin{array}{l}\text{compositions,} \\ \text{retracts,} \\ \text{pullbacks}\end{array}\right.$.

Proof for $^{\boxslash}K$:

__compositions:__ Let $f \boxslash K$ and $g \boxslash K$.

want a lift for

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[dd, "k"] \\ B \arrow[d, "g"'] & \\ C \arrow[r] & Y \end{tikzcd}$$

for all $k \in K$.

First, we have

$$\begin{tikzcd} A \arrow[rr] \arrow[d, "f"'] & & X \arrow[d, "k"] \\ B \arrow[r] \arrow[urr, dashed, "l"] & C \arrow[r] & Y \end{tikzcd}$$

because $f \boxslash k$. Then we have

$$\begin{tikzcd} B \arrow[r, "l"] \arrow[d, "g"'] & X \arrow[d, "k"] \\ C \arrow[r] \arrow[ur, dashed, "m"] & Y \end{tikzcd}$$

because $g \boxslash k$. $m : C \to X$ is now the desired solution.

Retracts

Say f is a retract of g, and $g \boxslash k$.

We want l:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "k"] \\ B \arrow[r] \arrow[ur, dashed, "l"] & Y \end{tikzcd}$$

We have $m : D \to X$ because $g \boxslash k$:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "g"] & A \arrow[r] \arrow[d, "f"] & X \arrow[d, "k"] \\ B \arrow[r, "n"'] & D \arrow[r] \arrow[urr, dashed, "m"] & B \arrow[r] & Y \end{tikzcd}$$

But now $m \circ n : B \to X$ is the desired lift.

Pushouts:

Say f is a pushout of g:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & C \arrow[d, "f"] \\ B \arrow[r] & D \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

And $g \boxslash k$. Then we have $m : B \to X$:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & C \arrow[r, "h"] \arrow[d, "t"] & X \arrow[d, "k"] \\ B \arrow[r] \arrow[urr, dashed, "m"] & D \arrow[r] & Y \end{tikzcd}$$

But by the universal property of a pushout, to the pair of maps

$m : B \to X$ and $h : C \to X$ there

corresponds (a unique) map $\ell : D \to X$.

That is our lift. ■

The statements for $K^{\boxslash}$ follow similarly (or by duality).

Weak Factorization Systems

Def A weak factorization system on $\mathcal{C}$ consists of a pair (L, R) with $L, R \subseteq \mathrm{Mor}(\mathcal{C})$ satisfying:

1) (factorization) Any map $f \in \mathrm{Mor}(\mathcal{C})$

factors as $$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\ell"'] & & B \\ & C \arrow[ur, "r"'] & \end{tikzcd}$$, with $\ell \in L$, $r \in R$

and

2) (lifting) $L \boxslash R$.

Model Category Axioms

Finally we can give the main definition:

Def. Let $\mathcal{C}$ be a category with all small limits and colimits.

A <u>model structure</u> on $\mathcal{C}$ consists of three classes of morphisms (C, F, W)

$$\left(\begin{array}{l} C: \text{"cofibrations", denoted } \rightarrowtail \\ F: \text{"fibrations", denoted } \twoheadrightarrow \\ W: \text{"weak equivalences", denoted } \xrightarrow{\sim} \end{array}\right)$$

satisfying the axioms:

1) (2 out of 3) for any diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & B \\ & C \arrow[u] \end{tikzcd}$$

<u>two</u> of the arrows are in W $\Rightarrow$ <u>the other</u> is in W.

(as is the case for isomorphisms!)

2) (weak factorization systems)

$(C, F \cap W)$ and $(C \cap W, F)$ form weak factorization systems.

Note: elements of $C \cap W$ are called trivial cofibrations ($\overset{\sim}{\rightarrowtail}$) and elements of $F \cap W$ are called trivial fibrations ($\overset{\sim}{\twoheadrightarrow}$).

Quick consequences

- C (cofibrations) and $C \cap W$ (trivial cofibrations) are closed under compositions, retracts, and pushouts.
- F (fibrations) and $F \cap W$ (trivial fibrations) are closed under compositions, retracts, and pullbacks.
- Every map $f : A \to B \in \mathrm{Mor}(\mathcal{C})$ factors as

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, tail, "\text{cofib}"'] & & B \\ & C \arrow[ur, two heads, "\sim"', "\text{triv fib}"] & \end{tikzcd}$$

and also as:

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, tail, "\sim", "\text{triv cofib}"'] & & B \\ & C \arrow[ur, two heads, "\text{fib}"'] & \end{tikzcd}$$

Cofibrant / Fibrant replacement

There are special "well-behaved" classes of objects:

Def An object X is <u>cofibrant</u> if the map $\emptyset \to X$ from the initial object of $\mathcal{C}$ is a cofibration.
An object Y is <u>fibrant</u> if the map $Y \to *$ to the terminal object is a fibration.

Thm Every object X has a <u>cofibrant replacement</u>, i.e. a weak equivalence

$$X^{cof} \xrightarrow{\sim} X$$

from a cofibrant object X^{cof}, and every object Y has a <u>fibrant replacement</u>, i.e. a weak equivalence

$$Y \xrightarrow{\sim} Y^{fib}.$$

Proof Factor
$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \\ X^{cof} \arrow[ur, "\sim"] \end{tikzcd}$$
, and

Factor
$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\sim"] & * \\ Y^{fib} \arrow[ur] \end{tikzcd}$$
.

Examples

- The category Top of compactly generated Hausdorff spaces has the Serre model structure in which

$$C = \{\text{(retracts of) relative CW-complexes}\}$$

$$F = \{\text{Serre Fibrations}\}$$

$$W = \{\text{weak equivalences}\}$$

Cofibrant objects = {(retracts of) CW-complexes}

Fibrant objects = {everything}

- The category $Ch_{\geq 0}$ of chain complexes has the projective model structure

$$C = \{\text{injective chain maps with projective cokernel}\}$$

$$F = \{\text{surjective chain maps in positive degree}\}$$

$$W = \{\text{quasi-isomorphisms}\}$$

Cofibrant objects = {complexes of projectives}

Fibrant objects = {everything}

What about homotopies?

Def. For X an object in a model category, a cylinder object $Cyl(X)$ is an object fitting into the factorization

$$\begin{tikzcd} X \sqcup X \arrow[r, "\text{fold}"] \arrow[dr, "\sim"'] & X \\ & Cyl(X) \arrow[u, "\sim"'] \end{tikzcd}$$

of the fold map $X \sqcup X \to X$.

A path object $path(X)$ is an object fitting into the factorization

$$\begin{tikzcd} X \arrow[r, "\text{diag}"] \arrow[dr, "\sim"'] & X \times X \\ & path(X) \arrow[u] \end{tikzcd}$$

of the diagonal map $X \to X \times X$.

(In Top the cylinder object is the cylinder and the path object is the path space).

Def. Given two maps $f, g : X \to Y$ we say f is left homotopic to g

$(f \sim_\ell g)$ if the map $(f,g): X \sqcup X \to Y$ extends to a map $h: \mathrm{Cyl}(X) \to Y$:

$$\begin{tikzcd} X \sqcup X \arrow[r, "{(f,g)}"] \arrow[d, tail] & Y \\ \mathrm{Cyl}(X) \arrow[ur, dashed] & \end{tikzcd}$$

we say f, g are <u>right homotopic</u> $(f \sim_r g)$ if the map $(f,g): X \to Y \times Y$ factors through $\mathrm{Path}(Y)$:

$$\begin{tikzcd} X \arrow[r, "{(f,g)}"] \arrow[d, dashed] & Y \times Y \\ \mathrm{Path}(Y) \arrow[ur, two heads] & \end{tikzcd}$$

<u>Theorem</u> If X is <u>cofibrant</u> and Y is <u>fibrant</u>, then left and right homotopy define equivalence relations on $\mathrm{Hom}(X, Y)$ that coincide, and equivalence is preserved by composition, and $\mathrm{Hom}(X,Y)/\sim$ is preserved by weak equivalence.

This gives us a way of getting a handle on the homotopy category:

Theorem: Let $\mathcal{C}^{cf}$ be the full subcategory of $\mathcal{C}$ on the cofibrant-fibrant objects.

Then $\mathcal{C}^{cf}/\sim \;\cong\; \mathcal{C}[W^{-1}]$ $(= \mathrm{Ho}\,\mathcal{C})$

ie on these special objects we can get the homotopy category just by quotienting by the homotopy equivalence relation.

Finally, we can obtain the universal functor $\mathcal{C} \to \mathrm{Ho}\,\mathcal{C}$ by precomposing with fibrant and cofibrant replacement functors so

$$\mathrm{Hom}_{\mathcal{C}}(QRX, QRY)/\sim \;\cong\; \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, Y).$$