Scissors congruence :

1900 Hilbert $\rightsquigarrow$ 23 problem

3^{rd} : s.c

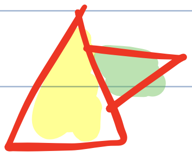
Fix n, $\mathbb{E}^n$, n-dim non-degenerate polytopes, $P, Q \cdots$

$P \sim_{s.c} Q$ if $P = \sum_i P_i$ st $g P_i = Q_i$
$Q = \sum_i Q_i$

$g \in G \trianglelefteq \mathrm{Ism}(\mathbb{E}^n)$

$P = \bigsqcup P_i$ st $\mu(P_i \cap P_j) = 0$

eg :-



$\sim_{s.c}$ is equivalence rel^n.

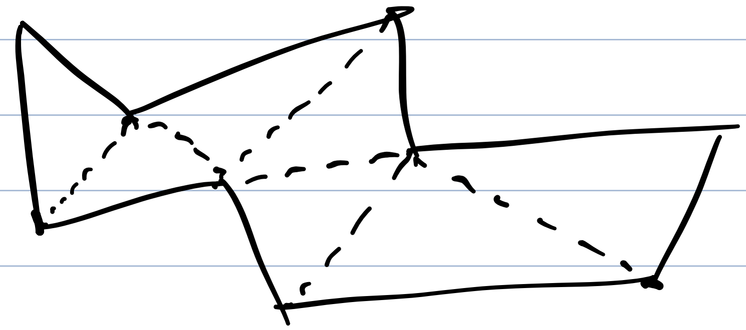
Rmk:- If $P \simeq_{s.c} Q$, then $\mathrm{vol}(P) = \mathrm{vol}(Q)$.

- In $\mathbb{E}^3$, given two polyhedra of same volume, are they s.c ?

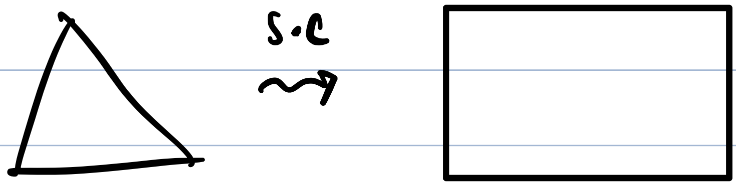
[1800 Wallace] If $\mathbb{E}^2$, $P \simeq_{s.c} Q$ iff $\mathrm{area}(P) = \mathrm{area}(Q)$
1833 Bolyai
1835 Gerwien

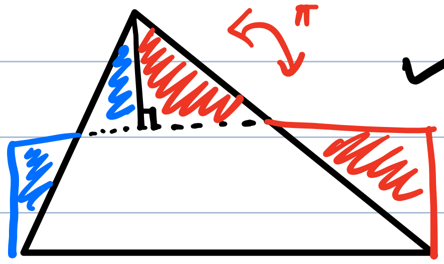
Pf: ($\Rightarrow$) easy.

($\Leftarrow$) Any P is s.c to a cube of same area.



P $\rightsquigarrow$ bunch of Δ's





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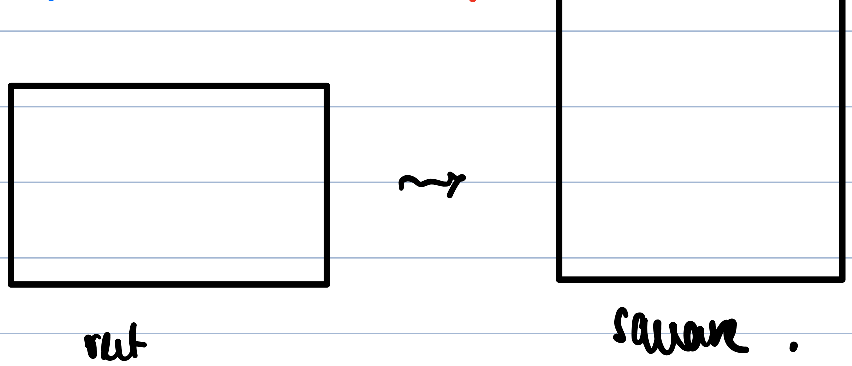

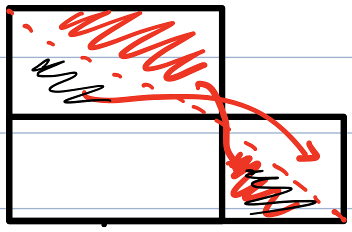

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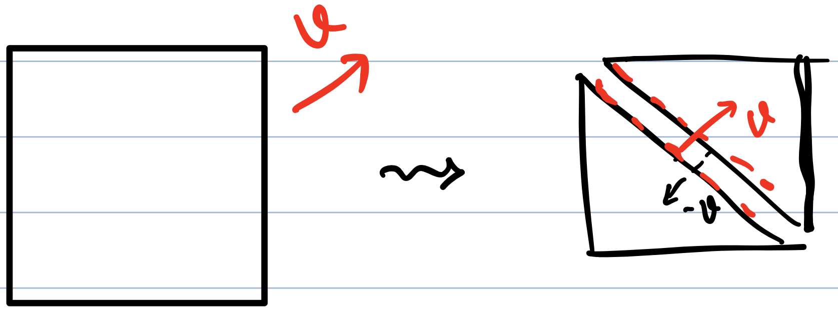
▨

Rmk:- s.c grp of polytopes in $\mathbb{E}^2$ with $G = \{\text{translation}, \text{rot by } \pi\}$ characterized by area.

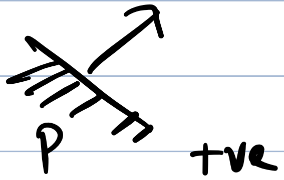
- $G = \{\text{translation}\}$

is area the only invariant of s.c in $\mathbb{E}^2$? No.

Thm [Hadwiger- Glur, 1951]. In $\mathbb{E}^2$, translation s.c is characterized by area + Hadwiger invariant.



$\forall v \in \mathbb{R}^2$, $L_v(\square) = \sum$ signed length's $\perp$ to v.

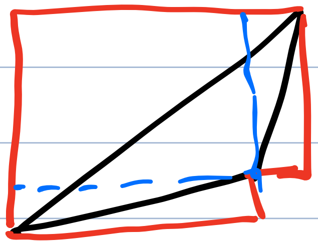


$\forall v$, L_v is a map on the t.s.c

Pf:- Let me prove: $\forall v, L_v(P)=0$, $P \underset{t.s.c}{\simeq} \square$

$P \rightsquigarrow$ bunch of Δ's

add Δ's + $\Delta \rightsquigarrow$ bunch of $\square$'s



$$\Delta + (\text{bunch of } \Delta\text{'s}) = (\text{bunch of } \square\text{'s})$$

$$\wr$$

$$= (\text{bunch of } \square\text{'s})$$

$$L_v\big(\Delta + \underset{\substack{\wr \\ \text{righ } \angle\, \Delta\text{'s}}}{\underline{\underline{(\text{bunch } \Delta\text{'s})}}}\big) = L_v(\text{bunch of } \square)$$

Rmk:- vol, area, L_v are additive.

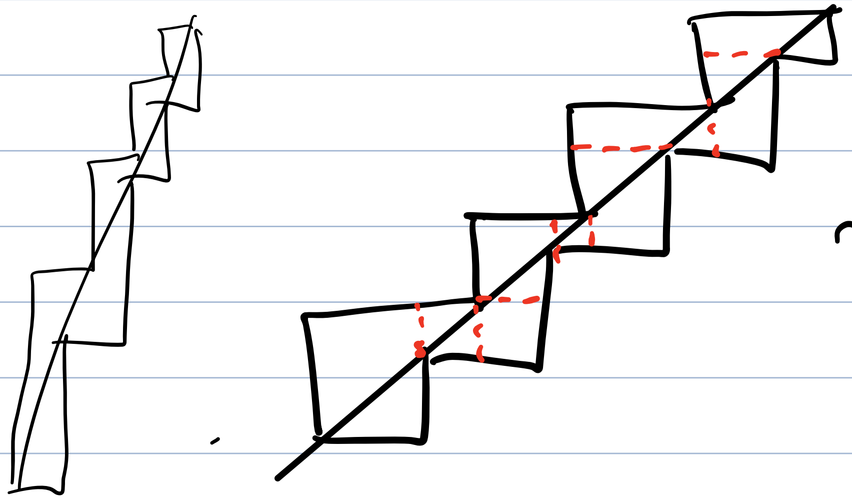
$P = \Delta_{p_i}$'s

$$L_v(\Delta) + \underline{\cancel{\text{dr}}\, L_v(\Delta)} = 0$$

$P + \Delta$'s $= \square$'s $\Rightarrow$ $0 + L_v(\Delta) = 0$

$L(P) + L(\Delta\text{'s}) = L(\square)$

$\Rightarrow$ all the added Δ's have $L_v = 0$ $\forall v$.



$\rightsquigarrow$ bunch of squares.

$P + \square$'s $\cong_{t.s} \square$'s $\cong_{t.i} \square + \square$'s

$$P + \cancel{\underbrace{\text{bunch } \square}} \simeq_{t.s.c.} \square \simeq_{t.sc} \underset{\substack{\uparrow \\ \text{same} \\ \text{area as } P}}{\square} + \cancel{\underbrace{\text{bunch } \square}}$$

$$\underline{A} + \underline{B} \simeq_{t.sc} \underline{C} + \underline{B} \Rightarrow A \simeq_{t.sc} C$$

- $s.c(\mathbb{E}^2, \mathrm{Isom}(\mathbb{E}^2)) = s.c(\mathbb{E}^2, \{\text{trans}, \overset{\text{rot}}{\pi}\}) \simeq \mathbb{R}$

  $s.c(\mathbb{E}^2, \{\text{trans}\})$ characterized by $\underset{\mathbb{R}}{\underset{\sim}{\text{vol}}}$, $\underset{\sim}{\text{Hadwiger inv.}}$

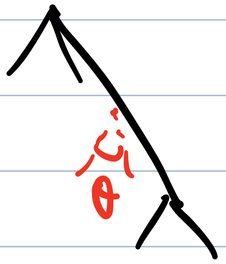
- In $\mathbb{E}^3$, all isom$(\mathbb{E}^3)$

Dehn (1901) : finds an invariant on s.c grp.

take any 3-polytope, P

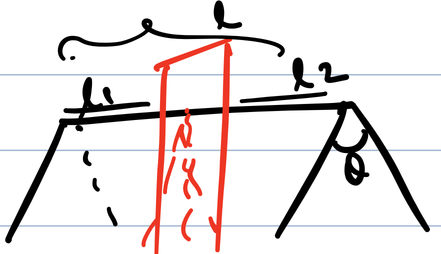
$$D(P): P \longmapsto \overset{\mathbb{R}\otimes_{\mathbb{Z}}\mathbb{R}/\pi\mathbb{Z}}{\sum_{e\in E} \ell(e)\otimes\theta_e}$$

E = set of edges.



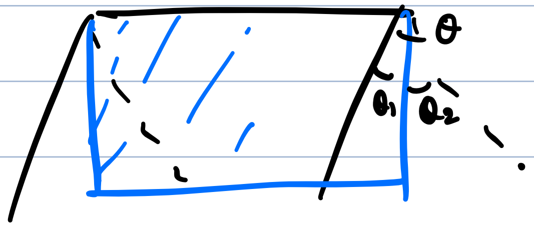
D is s.c invariant:

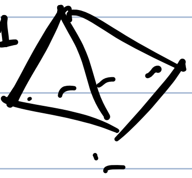
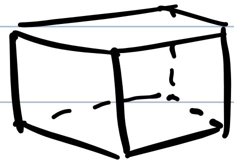
- $D(gP) = D(P)$
- $P = \sum P_i$, $D(P) = \sum_i D(P_i)$



$l_1\otimes\theta + l_2\otimes\theta = l\otimes\theta$

$l\otimes\theta_1 + l\otimes\theta_2 = l\otimes\theta$



[Dehn] : $\mathcal{D}($  $) \neq \mathcal{D}($  $)$

$\rightsquigarrow$ same volum

$$\mathcal{D}(\text{tetra}) = 6(1 \otimes \underbrace{\cos^{-1}(1/3)}_{\text{not a multiple of } \pi \ \& \text{ irr.}}) \neq 0$$

$$\mathcal{D}(\text{cube}) = 12(a \otimes \pi/2) = a \otimes 6\pi = 0$$

Q) Is vol + Dehn enough to characterize s.c ?

[1965] Sydler : Yes (Geometry)

[1968, 1972] Jessen : Yes (Algebraically, Polytope algebra)

Generalizes this to $\mathbb{E}^4$

for $n > 4$, open.

[1970s] [Jessen & Thorup : for t.s.c in $\mathbb{E}^n$ $\forall n$, vol + gen. Hadwiger inv characterize t.s.c .

- $[P + Q] = [P] + [Q]$ $\qquad$ s.e grp $= \dfrac{\text{gen by } [P]}{[P]+[Q]=[P+Q]}$
  $[P] = [gP]$.

  $\Big\downarrow$ algebra / K-theory of a ring.

- $R \in$ Ring, $\quad \mathbb{P} =$ monoid of isom classes of f.g proj modules $(\oplus)$

$\mathbb{R}$ $\qquad$ $\mathbb{P} =$ monoid of isom classes . $\qquad$ $\|$ finit dim v.sp.

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad$ $\|$ dimension.

$*$ $\underset{=}{K_0(R)} := \underset{\text{grp completion}}{\overset{=\mathbb{N}}{P^{-1}P}}$. $= \dfrac{\text{generated of } [Q]}{[P \oplus Q] = [P]+[Q]}$, $P, Q \in$ f.g proj R-mod

s.e

$0 \to P \to P \oplus Q \to Q \to 0$

$0 \to P \to P' \to P'' \to 0$

- $\underline{K_1(R)} := GL(R)^{ab}$ $\qquad$ $GL(R) = \mathrm{colim}\, GL_1(R) \hookrightarrow GL_2(R) \to Gl_3(R) \hookrightarrow \cdots$
  $\qquad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$
  $\qquad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$

$*$ $\downarrow$ failure of row-column reduction to I

$= \dfrac{GL(R)}{[GL(R), GL(R)]}$ $\qquad = \bigcup_n GL_n(R)$

$= \dfrac{GL(R)}{E(R)} \rightsquigarrow$ elementary matrices $\quad \begin{pmatrix} 1 & \ddots & \overset{r}{0}\,0 \\ 0 & & \ddots\; 1 \end{pmatrix}$

- $K_2(R) :=$ in terms of Steinberg grp. $\ldots$

$K_3(R) = ?$

$\vdots$

$K_n(R) = ?$

- (Quillen in 60's 70's)

$R \in$ Ring,

$$\begin{tikzcd} BGL(R) \arrow[d, "f"] \\ BGL(R)^+ \end{tikzcd} \qquad \begin{cases} \pi_1 = G \\ \pi_i = 0 \quad i>1 \end{cases}$$

st $F(f)$ is acyclic (homology of a pt).
& $\pi_1 F(f) = E(R)$

- $\pi_1(BGL(R)^+) = \pi_1(BGL(R)) / \pi_1 F(f) = Gl(R) / E(R)$
$= K_1(R)$.

- $\pi_0(BGL(R)^+) = 0$

- $\pi_2(BGL(R)^+) = K_2(R)$

$K(C) = K_0(R) \times BGL(R)^+$
$K_n(C) := \pi_n(K(C)) \quad \forall\ o$

- (Quillen) $R \rightsquigarrow \underline{\underline{\text{Proj}}}(R)$ obj : P f.g proj R-mod
mor: $P \to Q$ R-mod maps.

$\rightsquigarrow$ Abelian category .

$\rightsquigarrow$ SES

$0 \to P \xrightarrow{f} P' \xrightarrow{g} P'' \to 0$.

ses . ker g = img f , kerf = 0
cokerg = 0

Exact categories : $(\mathcal{C}, \mathcal{E})$

$\mathcal{C}$: additive category.

$\mathcal{E}$: set of $0 \to A \rightarrowtail B \twoheadrightarrow C \to 0$

Suppose. $\mathcal{C} \hookrightarrow \mathcal{A}$

$\uparrow$ abelian cat.

- $\mathcal{E}$ are precisely ses in $\mathcal{A}$
- suppose $A, C \in \mathcal{C}$ & $0 \to A \to B \to C \to 0$ ses in $\mathcal{A}$ then $B \in \mathcal{C}$

$K(\mathcal{C}) := \Omega B(Q\mathcal{C})$

obj : same as $\mathcal{C}$

mor : $P \longrightarrow Q$

$\cong$

eq. class $P \leftarrowtail R \twoheadrightarrow Q$

lemma $\mathcal{C} = \underline{\text{Proj}}(R)$

$\pi_1(B(Q\mathcal{C})) \cong K_0(R)$

$\pi_2(BQC) \cong K_1(R)$

$\pi_3($ " $) \cong K_2(R)$.

$\pi_n(K(\mathcal{C})) \cong \pi_n(BGL(R)^+) \;\; \forall \; n > 1.$

- $R \rightsquigarrow BGL(R)^+ \overset{\pi_n}{\rightsquigarrow}$

$\vdots$

- Exact category $\mathcal{C} \rightsquigarrow K(\mathcal{C})$

- $S^{-1}S$ const [Quillen] S is symm monoidal cat
- $S_\bullet$-const [Waldhausen]   Waldhausen cat
  $\cup\!|$
  Exact cat
- Segal's const ‖ Group completion.

Talk 2 :

- Segal's Construction :

$A \in$ Top group , $BA \in$ Top-grp .

$A, BA, B^2A, \ldots \rightsquigarrow$ Spectrum

$\downarrow$ coh thy

h^*

$\Gamma =$ FinSet$_*$

Γ-space, A is a Γ space if $n \in \mathbb{N}$, $A_n \in$ Top

$p_n : A_n \xrightarrow{\simeq} A_1 \times \cdots \times A_1$ $A(\{1\})$

$m_n : A_n \longrightarrow A_1$ $A : \Gamma^{op} \to$ Top .

$A \in \Gamma$-space ., $BA \in \Gamma$-space . $BA : \Gamma^{op} \to$ Top .

$(A)_1, (BA)_1, (B^2A)_1, \ldots$ $\longleftarrow$ $\Sigma(BA) \longrightarrow B^{n+1}(A)$

$\downarrow$ H-space , $\downarrow$ H-space inverse

. . . .

$\Omega B(A_1), BA, B^2A, \ldots \rightsquigarrow$ Spectrum.

$\downarrow$

generalized coh theories .

- Given any topo M , $\Omega BM \rightsquigarrow$ "group completion"

  $\pi_0(M)$.. $\pi_0(\Omega BM)$ is a grp.

- e.g:- Symm monoidal cat, $(\mathcal{C}, \otimes)$

$$\begin{tikzcd} \underline{\underline{\mathrm{Proj}}}(R) , \underline{\underline{\mathrm{Free}}}^{\mathrm{isom}}(R) \arrow[r] & R \arrow[loop above, "GL(R)"] & R^2 \arrow[loop above, "GL_2(R)"] & R^3 \arrow[loop above, "GL_3(R)."] \end{tikzcd}$$

$$B\underline{\underline{\mathrm{Free}}}^{\mathrm{isom}}(R) = |N_\cdot(\underline{\underline{\mathrm{Free}}}^{\mathrm{isom}}(R))| = \bigsqcup_{n\geq 1} BGL_n(R)$$

$$B\underline{\underline{\mathrm{Proj}}}^{\mathrm{isom}}(R) = |N_\cdot(\underline{\underline{\mathrm{Proj}}}^{\mathrm{isom}}(R))| = \bigsqcup_{P} BAut(P)$$

$$K^{\mathrm{segal}}(C) := \Omega B(\bigsqcup_P BAut(P)) \simeq \Omega B(\bigsqcup_{n\geq 1} BGL_n(R))$$

Fact: $BGL(R)^+$ is a H-space & it's universal among all $\overset{\text{maps from } BGL(R)\text{ to}}{\wedge}$ H-spaces that kill $E(R)$.

$$\begin{tikzcd} \overset{M\quad N}{B(Aut(P) \times Aut(Q))} \arrow[r] & \overset{\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}}{BAut(P \oplus Q)} \end{tikzcd}$$

$$\begin{tikzcd} BGL(R) \arrow[rr] \arrow[dr] & & \Omega B(\underbrace{\bigsqcup_P BAut(P)}_{\text{monoid}}) \\ & BGL(R)^+ \arrow[ur, "\simeq"'] & \end{tikzcd}$$

Segal-Mcduff grp completion thm to show $H_*(\ ) \cong H_*(\ )$

Whitehead to show $\Rightarrow$ wk eq $\Rightarrow$ h.e.

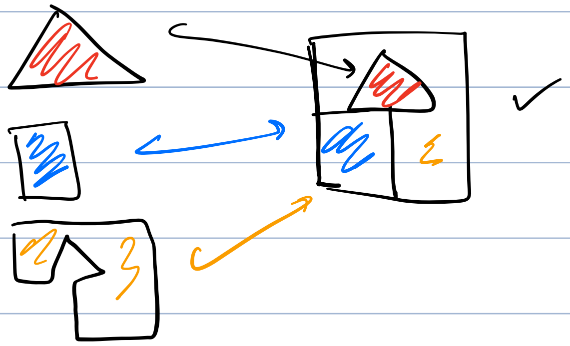
Assemblers CatFam.

weak Assemblers [Zakh 2017, BGMMZ 2024].

$\mathcal{C}$, with a collection of covering families.

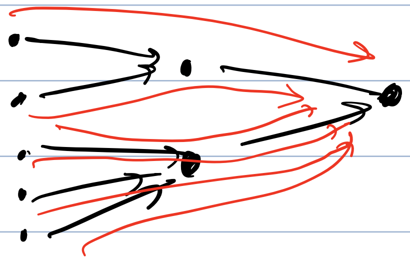
$$\{f_i : A_i \longrightarrow B\}_{i\in I}$$

eg :-



$$\{\forall i, U_i \longrightarrow X\}$$

- $\{A \xrightarrow{\mathrm{Id}} A\}$ $\forall A$ covering.
- composition of coverings.



- $*$ base pts $\mathcal{C}(*,*)=\mathrm{Id}_*$, $\mathcal{C}(c,*)=\emptyset$ $c\neq *$.
- $\{* \xrightarrow{\mathrm{Id}} *\}_{i\in I}$ covering.

$\mathcal{C}$ with covering family str is a wk assembler if

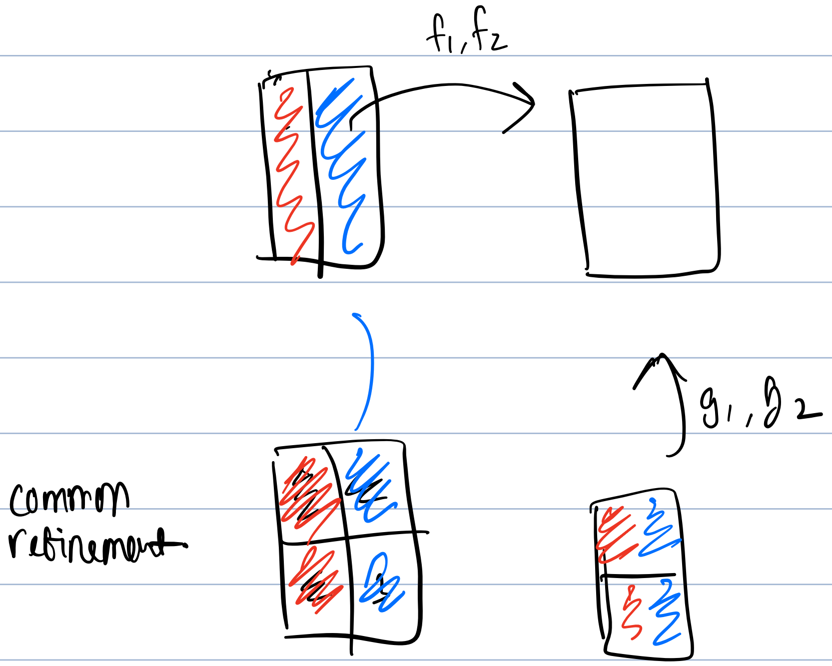
(I) $\mathcal{C}$ has initial obj, $\emptyset$, $\emptyset = *$

(D) all coverings are disjoint coverings.

$$\{A_i \xrightarrow{f_i} A\}_{i \in I}$$

$$\begin{tikzcd} \emptyset = A_i \times_A A_j \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A_j \arrow[d] \\ A_i \arrow[r] & A \end{tikzcd}$$

(R) Any two coverings of same obj have a common refinement.



(M) all morphisms in $\mathcal{T}$ are monomorphisms.

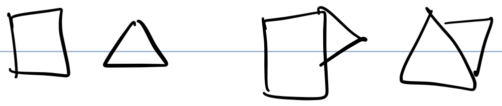
e.g :-
- Assembler
- $G = \ast \circlearrowleft G \qquad \{\ast \xrightarrow{g} \ast\}$
- $A \in Ab$ with discrete topology.

  $\mathcal{E}A$ obj : A

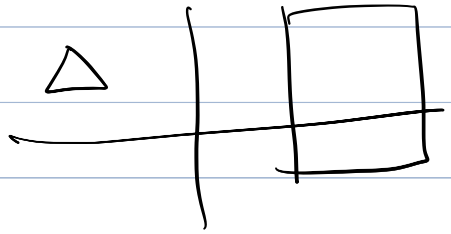
  mor : $a, b \in A \quad a \xrightarrow{\exists!} b$

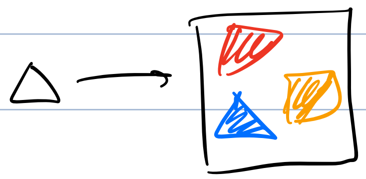
  $$\{a_i \to a\}_{i \in I} \quad \text{iff} \quad \sum_i a_i = a$$
- def $X = \mathbb{R}^n, H^n, S^n$ & $G \leq \mathrm{Isom}(X)$.

Polytope Category : P^{x}_{G} obj : polytopes : finite union of non-deg n-simplices .



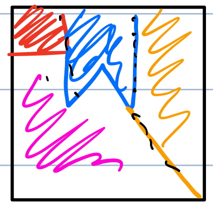
mor : $P \xrightarrow{g} Q$ if $gP \hookrightarrow Q$





covering family : $\{g_i : P_i \to P\}_{i \in I}$ & their pairwise intersection $\mu(P_i \cap P_j) = 0$.

& $\bigcup_i P_i = P$.

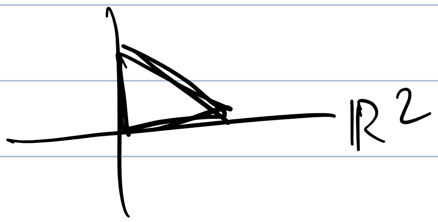


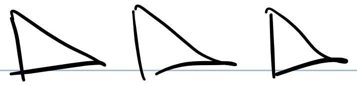
Category of covers : $W(\mathcal{C})$

objs : $P = (I, \{P_i\}_{i \in I})$ $P_i \in$ obj $\mathcal{C}$.

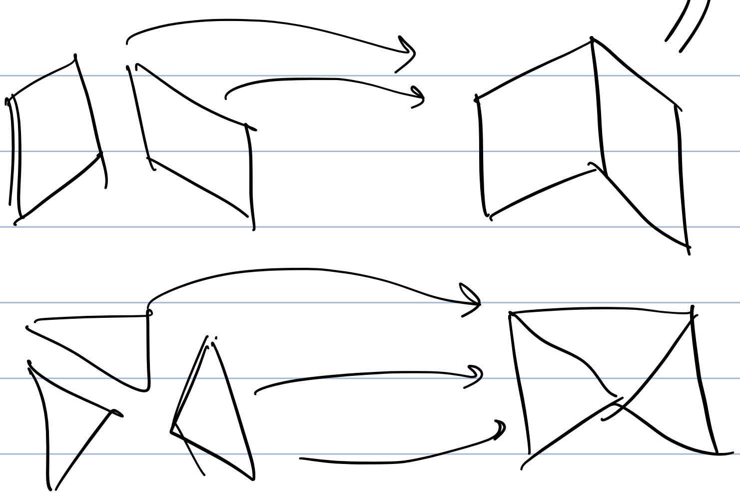
↙

formal copies of the same polytope





mor : $P = (I, \{P_i\}_{i \in I}) \longrightarrow Q = (J, \{Q_j\}_{j \in J})$



how to break a polytope.

Rmk:- $W(\mathcal{C})$ is a symm monoidal category.

$$(I, \{P_i\}_{i\in I}) \sqcup (J, \{Q_j\}_{j\in J})$$

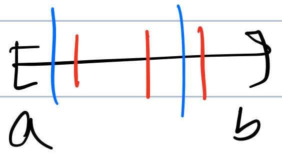
$$:= (I \sqcup J, \{P_i\}_{i\in I} \cup \{Q_j\}_{j\in J})$$

e.g:- $P^{[0,1]}_{\text{inclusions}}$. $[0,1]$

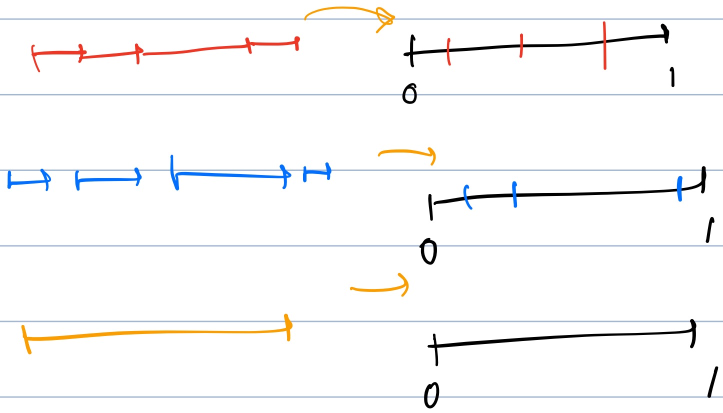
obj $[a,b]$ st $0 \leq a \leq b \leq 1$

mor $[a,b] \hookrightarrow [c,d]$

covering maps



$W(P^{[0,1]}_{inc})$



Define :- $K(\mathcal{C}) \underset{\substack{\uparrow \\ \text{wk assembler.}}}{}$ = Segal's const on $W(\mathcal{C})$

$$= \Omega B(|N_\bullet W(\mathcal{C})|)$$

$$K_n(\mathcal{C}) := \pi_n(K(\mathcal{C}))$$

e.g :- $K_0\left(P^{R^2}_{\mathrm{Isom}(R^2)}\right) = \mathbb{R}$ area.

$$\text{vol}: \; P^{R^2}_{\mathrm{Isom}(R^2)} \longrightarrow \underset{\circlearrowleft}{\mathcal{E}\mathbb{R}} \qquad \begin{array}{l} \text{obj} \;\; \mathbb{R} \\ \text{mor} \;\; \exists!\; r_1 \xrightarrow{\exists!} r_2 \;, r_1, r_2 \in \mathbb{R} \end{array}$$

G acts trivially.

$$P \longmapsto \text{vol}(P)$$

$$\{P_i \xrightarrow{f_i} P\}_{i \in I} \longmapsto \sum \text{vol}(P_i) \longrightarrow \text{vol}(P)$$

$$\begin{tikzcd}
K\left(P^{R^2}_{\mathrm{Isom}(R^2)}\right) \arrow[r] \arrow[d] & K(\mathcal{E}(\mathbb{R})) \arrow[d] \\
\text{S.C grp} = K_0(\quad) & K_0(\,) \cong \mathbb{R}
\end{tikzcd}$$

– G-S.C group of polytopes in $X \cong K_0(P^X_G)$

Category of weak Assemblers, $\mathcal{C}$

$$\begin{tikzcd}
\mathcal{C} \arrow[d, squiggly] \\
W(\mathcal{C}) \arrow[d, squiggly] \\
\mathcal{G}(C) = W(\mathcal{C})[W(\mathcal{C})^{-1}] \arrow[d, squiggly] \\
\text{groupoid.}
\end{tikzcd}$$

$W(\mathcal{C})^{-1}$ ← invert all arrows.

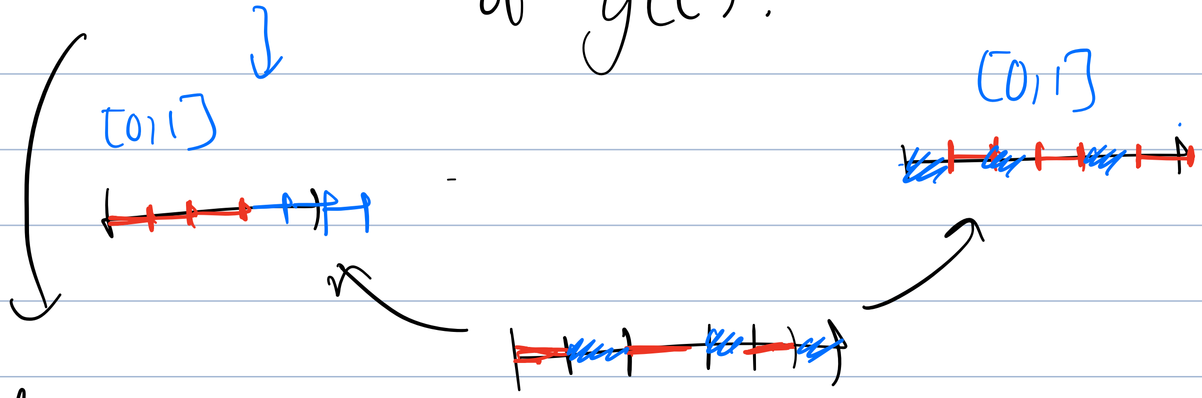
$$\begin{tikzcd}
\underline{P} = (I, \{P_i\}_i) & & (J, \{Q_j\}_{j\in J}) = Q \\
& (K, \{R_k\}_{k\in K}) \arrow[ul, "W(\mathcal{C})"] \arrow[ur, "\in W(\mathcal{C})"'] & \\
& \underline{R} \arrow[u, phantom, "\simeq"] &
\end{tikzcd}$$

$P \simeq_{s.c} Q$, $\mathcal{G}(C)$: symm monoidal cat , $\sqcup$

Thm [KLMMS, 2024]

$$K(\mathcal{C}) \simeq \Omega B(\underbrace{B\mathcal{G}(C)}) \qquad GL(R)$$

$\underset{s.c}{\mathrm{Aut}(P)}$:= automorphism of P in the category of $\mathcal{G}(C)$.



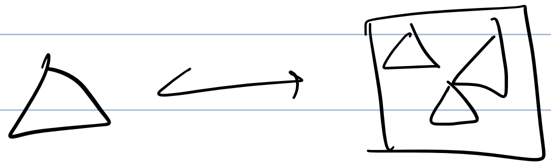
large group.

<u>Open</u> :- $\mathrm{Aut}(P) \cong \mathrm{Aut}(Q) \Rightarrow P \simeq_{s.c} Q$.

<u>Thm [KLMMS]</u>

- $K(P^{E^n}_{isom}) \cong K_0(P^{E^n}_{isom}) \times B\mathrm{Aut}(P)^+$ $\|$ $K(R) := K_0(R) \times BGL(R)^+$
- $H_*(\mathrm{Aut}(P)) \cong H_*(\mathrm{Aut}(Q))$ $\forall$ P, Q polytopes $P^{\times}_{G}$
  $\mathrm{vol}(P) \neq 0$
  $\mathrm{vol}(Q) \neq 0$.

not dependent on any choice.



$H_*(\mathrm{Aut}(P))$ is a ring.

$$\underset{\circlearrowleft\square}{\mathrm{Aut}(P)} \times \underset{\circlearrowleft\square}{\mathrm{Aut}(P)} \longrightarrow \mathrm{Aut}(\overset{\circlearrowleft}{P} \sqcup \overset{\circlearrowleft}{P})$$

$$\begin{tikzcd}
H_*(\mathrm{Aut}(P)) \times H_*(\mathrm{Aut}(P)) \arrow[r, "m"] & H_*(\mathrm{Aut}(P \sqcup P)) \arrow[d, phantom, "\cong" rotate=90] \\
& H_*(\mathrm{Aut}(P))
\end{tikzcd}$$

- $K_1(P^X_G) \cong \mathrm{Aut}(P)^{ab}$ $\quad\Big\|\quad$ $K_1(R) = GL(R)^{ab}$.

- $K_*(P^{E^n}_G)$ are rational [Malkiewich, 2022]

  __open__: (H^n, S^n)

- __open__: $K_1(E^2) = 0$

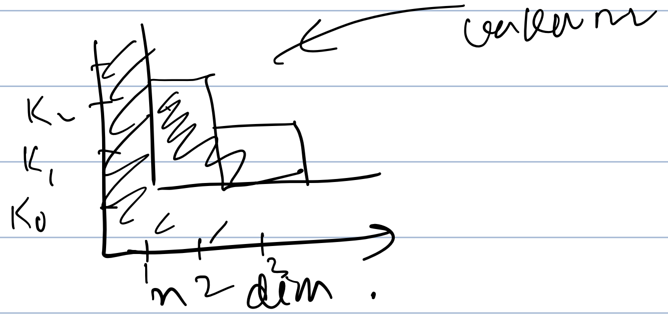
- __open__: can you describe Hadwiger invariants for K_1

- $K_n(P^X_{\underline{Isom(X)}}) \cong \mathbb{R}^{\wedge(n+1)}$, $\underline{X = E^1, S^1}$ (Markewich).

- can you prove any non-trivial fact about any other $X = E^n, S^n, H^n$, $n > 1$ (__open__) . ∎

  $\uparrow$ hyperbolic

$K_3(P^{S^2}_{Isom(X)}) = =$



Talk 3 (Dennis trace)

$R \in \mathrm{Ring}$

$$K(R) = BGL(R)^{+} \times K_0(R)$$

$$K_n(R) = \pi_n(K(C)) .$$

$HH_n(R)$:= Hochschild homology of ring R.

$$\mathrm{Dtr} : K_n(R) \longrightarrow HH_n(R).$$

Treat $GL_n(R) = * \circlearrowleft GL_n(R)$.

groupoid as all mor are isomorphism

$G \qquad * \circlearrowleft G$

$BG_n = (N_\cdot G)_n \qquad (g_1, \cdots, g_n)$

$(B^{cyc}G)_n = (N^{cyc}_\cdot G)_n \qquad (g_0, g_1, \cdots, g_n)$

$$\begin{tikzcd} B^{cyc}_\cdot \mathcal{C} \arrow[r, "\text{forgetting}"'] & B\mathcal{C} \arrow[l, dashed, bend right, "\mathcal{C}\text{ groupoid.}"'] \end{tikzcd}$$

$$\underset{*}{\overset{R}{\circlearrowleft}}$$

$$\begin{tikzcd} BGL_n(R) \arrow[r] & B^{cyc}GL_n(R). \arrow[r] & B^{cyc,\otimes}\underline{\underline{M_nR}}. \arrow[rr, bend right] & & B^{cyc,\otimes}\underline{\underline{R}} \arrow[ll, "\simeq"'] \end{tikzcd}$$

$$(g_1, \cdots, g_n) \longmapsto ((g_1 \cdots g_n)^{-1}, g_1, \cdots, g_n)$$

$$(\underline{g_0}, \cdots, g_n) \longmapsto \underline{g_0} \otimes \cdots \cdots \otimes g_n$$

$$(\underset{\in M_n(R)}{g_0} \otimes \cdots \otimes g_n) \mapsto (g_0 \odot \cdots \odot g_n)$$

$$(A \odot B)_{pq} = \sum_{i=1}^{n} A_{pi} \otimes B_{iq}.$$

$(B^{cyc,\otimes} R)_n = R \otimes \cdots \otimes R$ n+1 times.

$$\begin{tikzcd} R \arrow[r, shift left=2, leftarrow] \arrow[r, rightarrow] \arrow[r, shift right=2, leftarrow] & R \otimes_{\mathbb{Z}} R \arrow[r, shift left=4, leftarrow] \arrow[r, shift left=2, rightarrow] \arrow[r, leftarrow] \arrow[r, shift right=2, rightarrow] \arrow[r, shift right=4, leftarrow] & R \otimes R \otimes \cdots\cdots \end{tikzcd}$$

$\underline{HH_n(R)} = \pi_n(|B^{cyc,\otimes} R|)$

$$\bigsqcup_n BGL_n R \longrightarrow B^{cyc,\otimes} R.$$

$$\begin{tikzcd} \Omega B(\bigsqcup_n BGL_n R) \arrow[r, "\text{Dennis trace}"'] \arrow[dd, equal] & \Omega B(B^{cyc,\otimes} R). \arrow[d, "\wr\mathrm{S}"] \\ & B^{cyc,\otimes} R. \arrow[d, equal] \\ \underline{K(\mathcal{C})} & HH(R). \end{tikzcd}$$

Do this for assemblers.

Morita.

$$\begin{tikzcd} BGL_n(R) \arrow[r] & B^{cyc}GL_n(R) \arrow[r] & B^{cyc,\otimes}M_nR \arrow[r] & B^{cyc,\otimes}R \end{tikzcd}$$

$GL_n(R)$ groupoid.

invertible matrices to all matrices.

- **Assembler**: categories $\mathcal{C}$, with multimorphisms.

  covers.

$$\begin{tikzcd} P_1 \arrow[dr] & \\ P_2 \arrow[r] & P \\ \vdots & \\ P_n \arrow[uur] & \end{tikzcd}$$

disjoint : P_i & P_j

all mor in $\mathcal{C}$ are mono.

$W(\mathcal{C})$ obj $(I, \{P_i\}_{i \in I}) = P$

mor covers. $P \leftleftarrows$ category of covers.

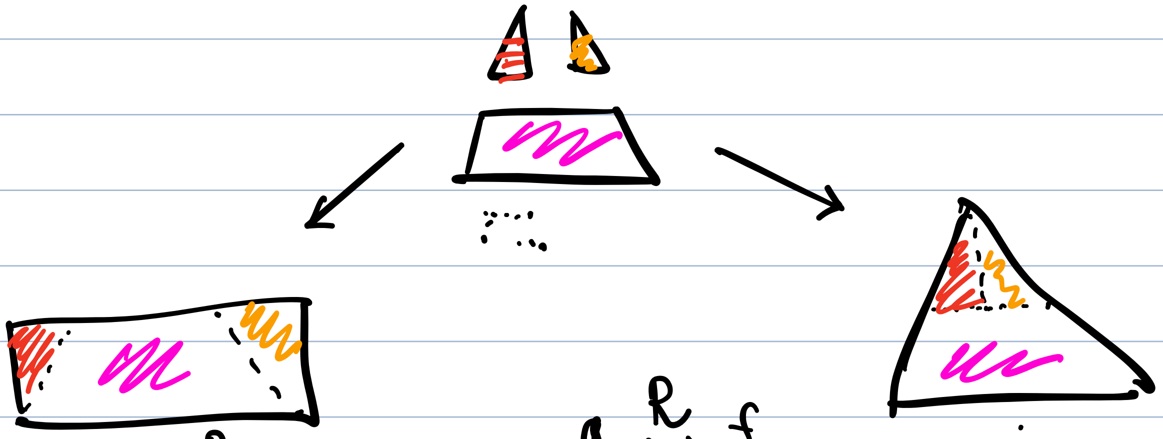
$G(\mathcal{C})$ obj $W(\mathcal{C})$

morphism.

$$\begin{tikzcd} & R \arrow[dl, "W(\mathcal{C})"'] \arrow[dr, "\in W(\mathcal{C})"] & \\ P & & Q \end{tikzcd}$$

$= (J, \{Q_j\}_{j \in J})$

$\mathbb{R}^2$



e.g:-
$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "g"] & \\ P & & Q \end{tikzcd}$$
inverse
$$\begin{tikzcd} & R \arrow[dl, "g"'] \arrow[dr, "f"] & \\ Q & & P \end{tikzcd}$$

Rmk:- $P, Q \in \mathcal{G}(\mathcal{C})$ $P \underset{s.c}{\simeq} Q$ iff $\mathrm{Mor}(P,Q) \neq \emptyset$

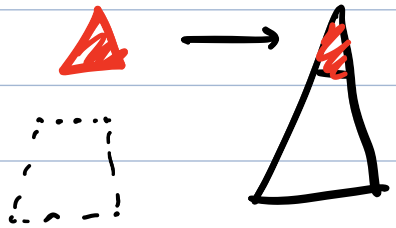
- Fact $\cdot K(W(\mathcal{C})) \simeq K(\underline{\mathcal{G}(\mathcal{C})})$

$N_\bullet \mathcal{G}(\mathcal{C}) \longrightarrow N_\bullet^{cyc} \mathcal{G}(\mathcal{C})$.

- Partial covers: $P \dashrightarrow Q$ $P, Q \in \mathrm{ob}\, W(\mathcal{C})$

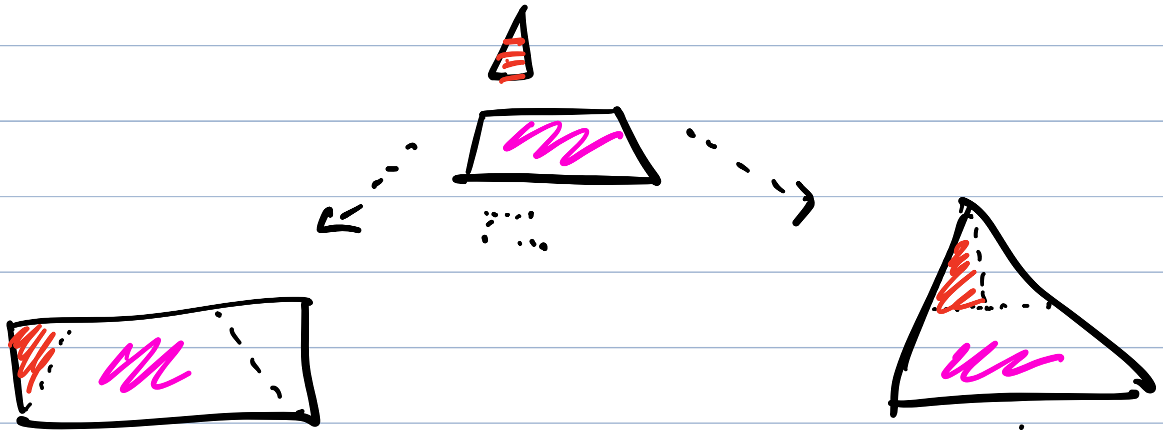
if $\exists P' \in \mathrm{ob}\, W(\mathcal{C})$ st

$$P \sqcup P' \xrightarrow[\in W(\mathcal{C})]{} Q$$



- $M(\mathcal{C})$ = Obj: obj $W(\mathcal{C})$

morphisms:
$$\begin{tikzcd} & R \arrow[dl, dashed] \arrow[dr, dashed] & \\ P & & Q \end{tikzcd}$$



$$\mathcal{G}(\mathcal{C}) \longrightarrow \mathcal{M}(\mathcal{C})$$

$$GL_n R \longrightarrow M_n R.$$

Dennis trace :

$$\begin{tikzcd}
N_\bullet \mathcal{G}(\mathcal{C}) \arrow[r] \arrow[d, squiggly, "\Omega B."] & N_\bullet^{cyc} \mathcal{G}(\mathcal{C}) \arrow[r] & N_\bullet^{cyc} \mathcal{M}(\mathcal{C}). \\
K(\mathcal{C}) \arrow[r] & \Omega B(N_\bullet^{cy} \mathcal{M}(\mathcal{C})) \arrow[d, equal] & \\
 & HH^{p}(\mathcal{C}) &
\end{tikzcd}$$

- Take enrichment of Hom set.

$$N_\bullet^{cyc} \mathcal{M}(\mathcal{C}) \longrightarrow N_\bullet^{cyc} \underset{\sim}{\mathbb{Z}\mathcal{M}}(\mathcal{C}).$$

$\mathcal{C} \rightsquigarrow \mathbb{Z}\mathcal{C}$ obj C.

mor. $\mathbb{Z}\mathrm{Hom}_{\mathcal{C}}(a,b) =: \mathbb{Z}\mathcal{C}(a,b).$

- **Morita**: $\mathbb{Z}M(\mathcal{C}) \longleftrightarrow \mathbb{Z}\underline{P}$. $P \in ob\, M(\mathcal{C})$

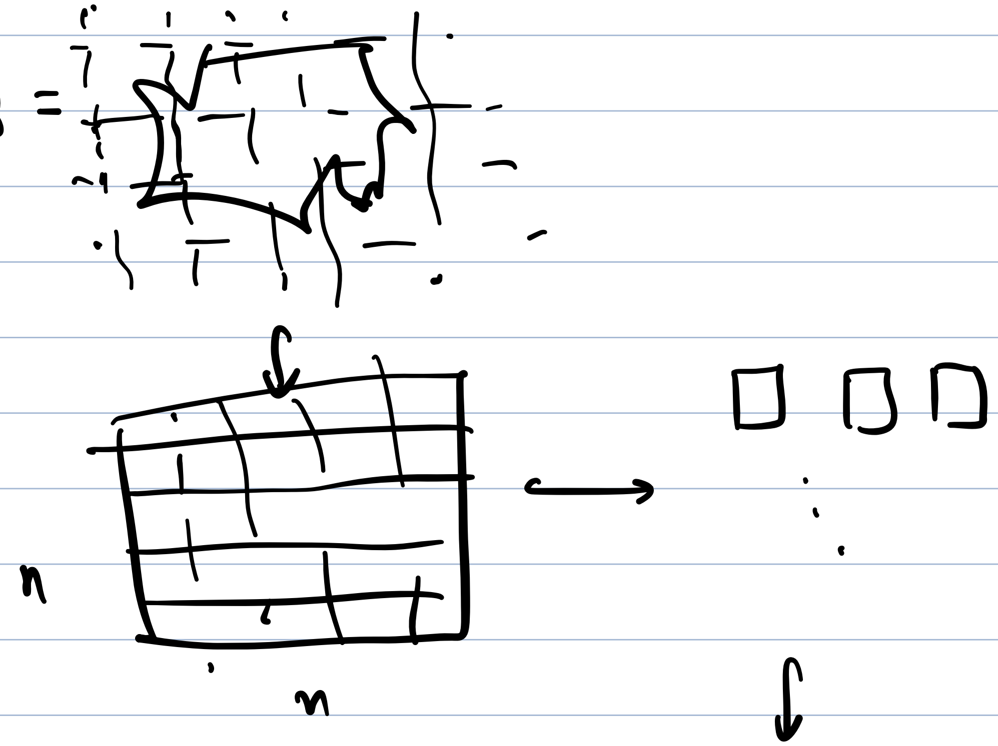
is morita equivalent if

$\forall Q \in obj\, M(\mathcal{C})$

$Q \longleftrightarrow P^{\sqcup n}$

eg:- $\mathcal{P}^{\mathbb{R}^2}_{isom}$. $P = \square$

$Q =$



$n^2 \square$

$= \square^{\sqcup n^2}$

Morita:- $\mathcal{C} \hookrightarrow \mathcal{D}$ every obj in $\mathcal{D}$ can be obtained by cofib., retracts.

$N^{cyc}_\bullet \mathbb{Z}M(\mathcal{C})) \simeq N^{cyc}_\bullet \mathbb{Z}\underline{P}$

$$\begin{tikzcd}
N_\bullet g(C) \arrow[r] \arrow[d, squiggly, "\Omega B"'] & N_\bullet^{cyc} g(C) \arrow[r] & N_\bullet^{cyc} \mathbb{Z}M(\mathcal{C}) & N_\bullet^{cyc} \mathbb{Z}\underline{P} \arrow[l, "\simeq"'] \arrow[d, squiggly] \\
K(\mathcal{C}) \arrow[rrr, "\text{Dennis trace.}"] & & & HH(R)
\end{tikzcd}$$

$\underline{P}$: obj P ⇝ monoid.

morphism
$$\begin{tikzcd}
& Q \arrow[dl, dashed] \arrow[dr, dashed] & \\
P & & P
\end{tikzcd}$$

$\mathbb{Z}P$: $\mathbb{Z}$ enrichment of P. ⇝ ring.

$$\begin{tikzcd}
N_\bullet^{cyc} \mathbb{Z}P \arrow[r, phantom, "\cong"] & N_\bullet^{cyc} R \\
& \mathbb{Z}\mathrm{Hom}_{M(C)}(P,P) \arrow[u]
\end{tikzcd}$$

$I=[0,1]$

e.g.:- $P^{I}_{\{1\}}$ obj: finite union of closed intervalls.

$[a_1,b_1] \cup [a_2,b_2] \cup \dots \cup [a_n,b_n]$

mor : inclusions.

$K_0(P^{I}_{\{1\}})$ $\left(\text{gen by } [\text{obj}] \big/ [P] = \sum [P_i]\right)$

- $K_0(P^{I}_{\{1\}}) \xrightarrow[\cong]{\phi} \bigoplus_{x\in(0,1]} \mathbb{Z} e_x$

$$[[a,b]] \longmapsto e_b - e_a$$

- $K_1(P^{I}_{\{1\}})$ · very large

Claim: $\mathbb{Z}\,\underline{\underline{I}} \hookrightarrow \mathbb{Z}\mathcal{M}(P^{I}_{\{1\}})$. is morita.

$$(\alpha, \{J_\alpha\})$$

$$\downarrow$$

$$(\alpha, I_\alpha) = I^{\sqcup(\alpha)}$$

$i=0,1$ $\quad K_i(P^{I}_{\{1\}}) \longrightarrow HH_i(R)$

$$R = \mathbb{Z}\,\underbrace{\mathrm{Hom}_{M(C)}(I, I)}.$$

$$\begin{tikzcd} & J \arrow[dl, dashed, hook', "inc"'] \arrow[dr, dashed, "inc"] & \\ I & & I \end{tikzcd} = \begin{tikzcd} J \arrow[d, dashed, "inc"] \\ I \end{tikzcd} = J$$

$\mathrm{Hom}_{M(C)}(I, I) \cong ob(P^{I}_{\{1\}})$ = finite union closed subset of I composite in Hom is intersection

$$\begin{tikzcd}
& & J\cap K \arrow[dl, dashed] \arrow[dr, dashed] & & \\
& J \arrow[dl, dashed, hook] \arrow[dr, dashed] & & K \arrow[dl, dashed, hook] \arrow[dr, dashed] & \\
I & & I & & I
\end{tikzcd} \equiv J \cap K$$

moniod is commutative.

$J \cap J = J$ ⇝ idempotent.

$\mathbb{Z}[\mathrm{Hom}_{M(C)}[I,I)] = R$ is commutative ring of basis elements are idempo

$\therefore$

$$HH_0(R) = R/_{[R,R]} = R$$

$$HH_1(R) \cong \Omega^1_{R|\mathbb{Z}}$$

$\downarrow$

generated by da, $a \in R$

$d(\lambda a + \mu b)$ $\quad$ $\mathbb{Z}$ linear.

$= \lambda da + \mu db$.

rel^n $\quad d(ab) = adb + bda$.

$$d(J) = d(J^2) = 2JdJ$$

multiply by J $\Rightarrow$ $JdJ = 2J^2dJ = 2JdJ$.

$\Rightarrow$ $JdJ = 0$

$$\Rightarrow dJ = 2\underset{\sim}{J\,d}J = 0$$

$$HH_1(R) = 0 .$$

$$\begin{tikzcd}
\bigoplus_{x \in [0,1]} \mathbb{Z} e_x \arrow[r] \arrow[dr] & R \arrow[d] \\
& \mathbb{R}
\end{tikzcd} .$$

$$H_0(\{1\}, \mathbb{R}) \cong R$$

$$\begin{tikzcd}
& 0 & \\
K_1(P^{I}_{\{1\}}) \arrow[ur] \arrow[rr] & & 0 \arrow[ul]
\end{tikzcd}$$

- trace map [BGMMZ]

$$tr : K_i(P^{X}_{G}) \longrightarrow \underline{\underline{H(G;A)}} \hookleftarrow G\text{-mod} .$$

$$H(\underset{\overset{\|}{G}}{\underline{\mathrm{Isom}(\mathbb{R}^1)}}, \mathbb{R})$$

$\mu : \operatorname{ob} P_G^X \longrightarrow A$ G-equiv.

$\{P_i \to P\}_i$ $\mu(P) = \sum_i \mu(P_i)$