Pseudofree Finite Group Actions on 4-Manifolds

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Theorem [Hambleton - Pamuk, 22, [19]]: If a finite group G acts pseudofreely, locally linearly and homologically trivially on a cl. conn, oriented 4 manifold with $\chi(M) \neq 0$, then

$$\text{rank}_p(G) \leq 1, \quad p \geq 5 \qquad \text{and} \qquad \text{rank}_p(G) \leq 2, \text{ for } p = 2, 3.$$

Theorem [Edmonds, 98]: $G \overset{\text{p. ll. ht}}{\curvearrowright}$ closed, simply-connected with $b_2 \geq 3$ then G is cyclic and acts semifreely.

- rank (G): (A way to measure a finite group)

p-rank: $\mathrm{rank}_p(G) = \max_r \{ (\mathbb{Z}_p)^r \leq G \}$.

$\mathrm{rank}(G) = \max_{p = \text{prime}} \mathrm{rank}_p(G)$.

Ex: $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$

$\mathrm{rank}_2(G) = 2$; $\mathrm{rank}_3(G) = 1$, $\mathrm{rank}(G) = 2$.

- $G \curvearrowright M$: $G \times M \longrightarrow M$ (i) $em = m$ (ii) $g(h(m)) = (gh)m$.

- Fixed point set of a subgroup: $K \leq G$

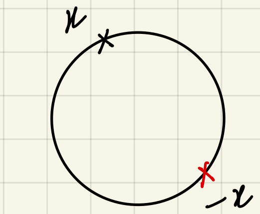
$$M^K = \{ x \in M \mid hx = x, \ \forall h \in K \}.$$

- Singular Set: $\Sigma = \bigcup_{K \neq e} M^K$

- **Free Action**

no fixed points

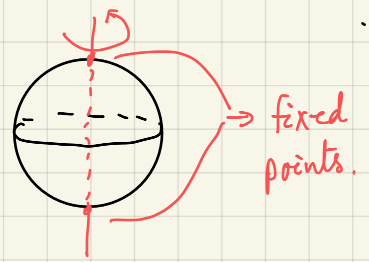
$\mathbb{Z}_2 \curvearrowright S^1$



Semifree Action

free action on a complement of a fixed set the whole group. ($\Sigma = M^G$)

$G = \mathbb{Z}_3$; $M = S^2$



Pseudofree Action

free action except for a discrete set.

(Σ is discrete.)

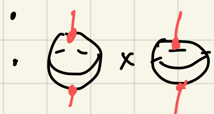
$G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, \gamma_1, \gamma_2, \gamma_3\}$

$M = S^2 \times S^2 \ni (u, v)$

$\langle \gamma_i \rangle \simeq \mathbb{Z}_2$.

$\gamma(u, v) = (\gamma u, \gamma v)$

- $\gamma = \gamma_i$;



- Homologically trivial: $G \curvearrowright M \rightsquigarrow$ induced action on $H^*(M)$ is trivial.

- Locally Linear: $G \curvearrowright M^4$, $\forall x \in M$, $\exists$ a G_x invariant nbd V_x s.t. $V_x \simeq \mathbb{R}^4$ and $G_x \curvearrowright V_x$ orthogonally.

G_x = isotropy subgroup.

[4, 24]

$$\chi(M) = \sum_{i \geq 0}^{\dim M} (-1)^i \dim H^*(M; \mathbb{Q})$$

Lets look at the following theorems once more:

Theorem [Hambleton - Pamuk, 22, [19]]: If a finite group G acts pseudofreely, locally linearly and homologically trivially on a closed, connected and oriented 4-manifold M with non-zero Euler Characteristic, then $\operatorname{rank}_p(G) \leq 1$ for $p \geq 5$ and $\operatorname{rank}_p(G) \leq 2$ for $p = 2, 3$.

$\operatorname{rank}_p(G) = 2$: $\quad p = 2, 3$: $\quad \mathbb{Z}_p \times \mathbb{Z}_p \leq G$

- $\mathbb{Z}_2 \times \mathbb{Z}_2 \curvearrowright M \rightarrow$
- $\mathbb{Z}_3 \times \mathbb{Z}_3 \curvearrowright M$
- p rank$=1$: minimal subgroups: cyclic, quaternion, metacyclic

Theorem [Edmonds, 98]: If a finite group G acts pseudofreely, locally-linearly and homologically trivially on a closed simply-connected 4-manifold with $b_2 \geq 3$ then G is cyclic and acts semi-freely.

(rank $= 1$)

The questions were :

- Can we eliminate rank 2 or rank 1 groups acting in this way?
- If some rank 2 or rank 1 do act in this way, can we put restrictions on M?

- Can we extend Edmond's result beyond simply-connected-ness?

$H_*(M; \mathbb{Z}_{(p)})$; $\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \mid p \nmid b \right.$

- $\mathbb{Z}_2 \times \mathbb{Z}_2 \curvearrowright M \longrightarrow$ M must have 2-local homology and intersection form of $S^2 \times S^2$

- $\mathbb{Z}_3 \times \mathbb{Z}_3 \curvearrowright M \longrightarrow$ M must have 3-local hom. of $\mathbb{C}P^2$.

- $\mathbb{Z}_p \times \mathbb{Z}_p \curvearrowright M \longrightarrow$ no action.
  $p \geq 5$

Action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on M^4 :

Tools: ① Cohomology of Finite groups [Kenneth Brown] :

Using Resolutions [5] — Using topology [2] → Alejandro-Adem.

Def^n : Let G be a group and R be a G-module. The Group Cohomology of G with coeff in R is

$$H^*(G;R) := H^*(K(G,1);R).$$

→ [Eilenberg 1947]

Def^n (Eilenberg – MacLane Space, $K(G,1)$) : Let G be a group. A CW-complex X is called an E.M Space of $K(G,1)$ for the group G if $\pi_1(X) = G$ and the universal cover of X is contractible.

$$K(\mathbb{Z},1) \cong S^1 \quad ; \quad K(\mathbb{Z}_2,1) \cong \mathbb{R}P^\infty.$$

Def^n : (Classifying space) The orbit space BG, of the universal principal G-bundle

$$G \hookrightarrow EG \longrightarrow BG \cong EG/G.$$

$$[X, BG] \xrightarrow{\simeq} \mathrm{Prin}_G(X)$$
$$f \longrightarrow f^*EG.$$

$$\begin{array}{ccc} f^*EG & \longrightarrow & EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BG \end{array}$$

Lemma : If G is a discrete group, then BG is a $K(G,1)$ space.
[Alejandro Adem].

$$H^*(G;R) = H^*(K(G,1);R) = H^*(BG;R).$$

Example: ① $G = \mathbb{Z}_2$.

$$\mathbb{Z}_2 \longrightarrow S^\infty \longrightarrow \mathbb{R}P^\infty$$

$$H^*(\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z}[u]/(2u) \; ; \quad u \in H^2(\mathbb{Z}_2;\mathbb{Z}) = \mathbb{Z}_2.$$

$$\| $$

$$H^*(\mathbb{R}P^\infty, \mathbb{Z})$$

$$\boxed{H^*(\mathbb{Z}_2;\mathbb{Z}_2) = \mathbb{Z}_2[x] \quad |x| = 1}$$

② $G = \mathbb{Z}_p$; $\quad H^*(\mathbb{Z}_p, \mathbb{Z}) = \mathbb{Z}(u)/(pu)$.

$p =$ prime.

③ $G = \mathbb{Z}_2 \times \mathbb{Z}_2$

$$H^*(G;\mathbb{Z}) = \frac{\mathbb{Z}[u_1, u_2](\mu)}{(2u_1, 2u_2, 2u, \mu^2 = u_1 u_2(u_1+u_2))}$$

$$u_1, u_2 \in H^2(G;\mathbb{Z})$$

$$\mu \in H^3(G;\mathbb{Z}).$$

Borel Construction :

G = finite group.

$G \longrightarrow EG \longrightarrow BG$ principal G-bundle

M = closed, connected, oriented G-manifold.

$M_G = M \times_G EG \equiv (M \times EG)/G$.

Associated Bundle :

$EG \xrightarrow{\pi} BG$ principal

$$\begin{array}{ccccc} M & \xrightarrow{p} & (M \times EG)/G \equiv M_G & \longrightarrow & BG \\ m & \hookrightarrow & (m, e) & \longrightarrow & \pi(e). \end{array}$$

Borel Cohomology :

$$H^*_G(M) := H^*(M_G)$$

$\underline{H^*_G}$ → cohomology functor.

For $g \in H^*(BG)$, $m \in H^*_G(M)$

$$g m = \pi^*(g) \cup m.$$

makes $H^*_G(M)$ a $H^*(BG)$ module.

[39, Tom Dieck].

Borel Spectral Sequence :

Consider the bundle $M \longrightarrow M_G \longrightarrow BG$.

The Lerray - Serre spectral sequence of the above bundle is called Borel spectral seq.

Theorem : Given $M \longrightarrow M_G \longrightarrow BG$ and a trivial action of $\pi_1(BG) = G$ on $H^*(M)$, $\exists$ a coho. seq.

$\{ E_r^{*,*}, d_r \}$ with E_2 - page.

$$E_2^{k,l} = H^k(G; H^l(M)) \Longrightarrow H^{k+l}(M_G) \equiv H_G^{k+l}(M)$$

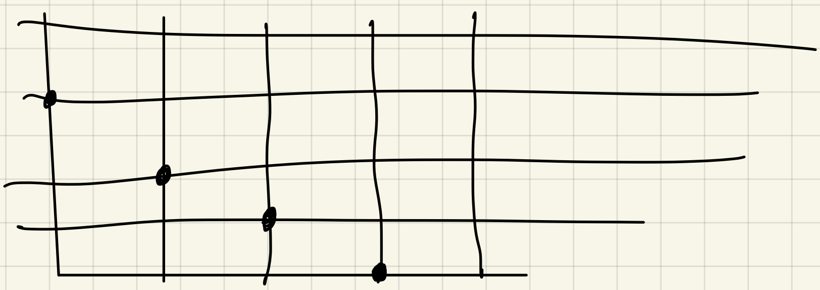
s.t. ① $d_r^{k,l} : E_r^{k,l} \longrightarrow E_r^{k+r,\, l-r+1}$

② $E_{r+1}^{k,l} = \operatorname{Ker} d_r^{k,l} \big/ \operatorname{Img} d_r^{k-r,\, l+r-1}$

- $H_G^n(M)$ admits a filtration

$$0 = F^{n+1,-1} \subset \underbrace{F^{n,0} \subset F^{n-1,1}} \subset \cdots \subset F^{0,n} = H_G^n(M).$$

$$F^{n-1,1} \big/ F^{n,0} = E_\infty^{n-1,1}$$



$$H_G^n(M) = \bigoplus E_\infty^{k,\, n-k}.$$