Pseudofree Finite Group Actions on 4-Manifolds

@ IISER - Kolkata
Feb 27, 2025

Feb 27 : Rank two finite group actions

$\downarrow$

$(G = \mathbb{Z}_2 \times \mathbb{Z}_2)$

$b_i'(M) = \dim_{\mathbb{Q}} H^i(M;\mathbb{Q})$

<u>Theorem</u>: Let M be a closed, connected and oriented 4-manifold with non-zero Euler Characteristic. If $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts pseudofreely, locally linearly and homologically trivially on M, then $b_2'(M) = 2$, $H_1(M;\mathbb{Z}_{(2)}) = 0$ and M must have the intersection form of $S^2 \times S^2$.

- p.l.h := pseudofree, locally linear and homologically trivial
- c.c.o := closed, connected and oriented manifold.
- Unless mentioned otherwise, G-action will be p.l.h and M will be c.c.o.

<u>Some results for $\mathbb{Z}_p \times \mathbb{Z}_p$ action, $p \geq 2$ prime.</u>

<u>Lemma 1</u>: Let $G = \mathbb{Z}_p \times \mathbb{Z}_p$, $p \geq 2$ prime.

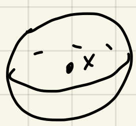
If $G \underset{plh}{\circlearrowright} M$ with $\chi(M) \overset{(>0)}{\neq 0}$, then

(i) G can't have a global fixed point.

and

(ii) each of the $(p+1)$-cyclic subgroups, $K \simeq \mathbb{Z}_p$ has $\chi(M)$ many fixed points.

proof: (i) · Suppose, $x \in M^{G}$

· Pseudofree $\Rightarrow$ x is isolated.  V_x^4

· V_x^4 is G_x invariant

$$\underbrace{\mathbb{Z}_p \times \mathbb{Z}_p = G \text{ acts freely on } \partial V_x^4 = S^3}_{\text{contradiction}}$$

— P.A. Smith's-p^2 cond.

(ii) $\mathbb{Z}_p \simeq K \leq G = \mathbb{Z}_p \times \mathbb{Z}_p$. Let $h \in K$.

$$\boxed{|M^K| = \chi(M)}$$

· Lefschetz number:

$$\Lambda_h = \sum_{i=0}^{\dim M} (-1)^i \, Tr\left(H^*(h) : H^i(M;\mathbb{Q}) \to H^i(M;\mathbb{Q}) \right)$$

$$\downarrow id_{H^*}$$

$$= \sum_{i=0}^{\dim M} (-1)^i \, Tr\left(id_{H^i(M;\mathbb{Q})} \right)$$

$$= \sum_{i=0}^{\dim} (-1)^i \dim H^i(M;\mathbb{Q})$$

$$= \chi(M).$$

• A variant of Lefschetz fixed point theorem ([9], [39, p.225])

$$|M^H| = |M^h| = \chi(M^h) \overset{\downarrow}{=} \Lambda_h = \underline{\underline{\chi(M)}} > 0.$$

$(\langle h \rangle = K)$ (p-free M^h is discrete, finite)

* **Theorem (Edmonds' Result, [9, 10, 32]):** Let M be a closed, connected and oriented G-manifold of dimension n. Let Σ be the singular set of the G-action. Then, we have the following isomorphism

$$\boxed{H^q_G(M) \cong H^q_G(\Sigma), \text{ for } q > \dim M.}$$

$$\Sigma = \bigcup_{e \neq K \underline{\underline{\leq}} G} M^K \quad ; \quad K \cong \mathbb{Z}_p.$$

---

[9] : Allan Edmonds, Aspects of Group actions on M^4, 1989.

[10] : ——————, Homologically Trivial group actions on 4-Man, 98

[32] : Michael McCooey, Symmetry group of 4-Manifolds, 2002

[39] : Tom Dieck, Transformation Groups, 1987

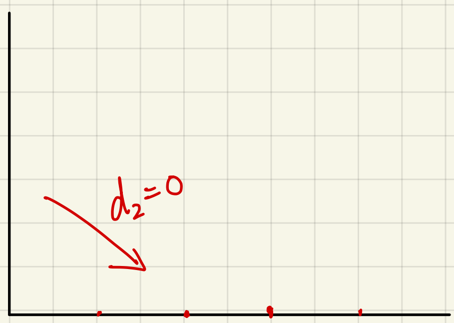
Using Edmonds' result, we prove the following important results.

$$G = \mathbb{Z}_p \times \mathbb{Z}_p \,, \quad p \text{ prime} \quad \underset{\text{p}\ell\text{h}}{\curvearrowright} \quad M^4 \ (\text{cc.o})$$

<u>Lemma 2:</u> $$\dim_{\mathbb{F}_p} H^q_G(M) = \begin{cases} 0, & q \geq 5 \text{ odd} \\ \frac{\chi(M)}{p}(p+1), & q > 5 \text{ even.} \end{cases}$$

<u>proof:</u> Edmonds' result $\longrightarrow$ $H^q_G(M) \cong H^q_G(\Sigma)$, $q > 4$

So, $$\dim H^q_G(M) = \dim H^q_G(\Sigma), \quad q > 4.$$
$$= \dim H^q(G, H^0(\Sigma))$$



$$E_2^{k,l}(\Sigma) = H^k(G, H^l(\Sigma))$$
$$= H^k(G, H^0(\Sigma)) = E_\infty^{k,0}.$$

- G does not have a global fixed point.
- $x \notin M^K \cap M^H$. $\quad K, H \leq G$.

M^K has $\chi(M)$ many fixed points, which are permuted in $\frac{\chi(M)}{p}$ orbits by G/K.

- $$H^0(M^K) = \bigoplus_{\chi(M)/p} \mathbb{Z}[G/K].$$

$$\left[\begin{array}{ll} G \curvearrowright G/K & \text{by left translation.} \\ \big\downarrow \text{extend} & \\ G \curvearrowright \mathbb{Z}[G/K] & \left(\sum_{[x]\in G/K} \lambda_x [x] \longrightarrow \sum_{[x]\in G/K} \lambda_x g[x] \right) \end{array}\right]$$

permutation $\mathbb{Z}G$ - module

So, $H^0(\Sigma) = \bigoplus_{K_i} H^0(\mathbb{Z}[G/K])^{\chi(M)/p}$.

$$\dim H^q_G(M) = \dim H^q_G(\Sigma)$$

$$= \dim \left(H^q(G, H^0(\Sigma)\right)$$

$$= \sum_{i=1}^{p+1} \dim H^q\left(G; (\mathbb{Z}[G/K_i])^{\chi(M)/p}\right).$$

$$= \frac{\chi(M)}{p} \sum_{i=1}^{p+1} \underbrace{\dim H^q(G; \mathbb{Z}[G/K_i])}_{\cong H^q(K_i, \mathbb{Z})} \quad [\text{Shapiro's Lemma}].$$

$$= \begin{cases} 0, & q \geq 5 \text{ odd} \\ \frac{\chi(M)}{p}(p+1); & q > 5 \text{ even.} \end{cases}$$

Lemma 3: $G = \mathbb{Z}_p \times \mathbb{Z}_p$, $p \geq 2 \curvearrowright M^4$.

In the integral Borel spectral seq.

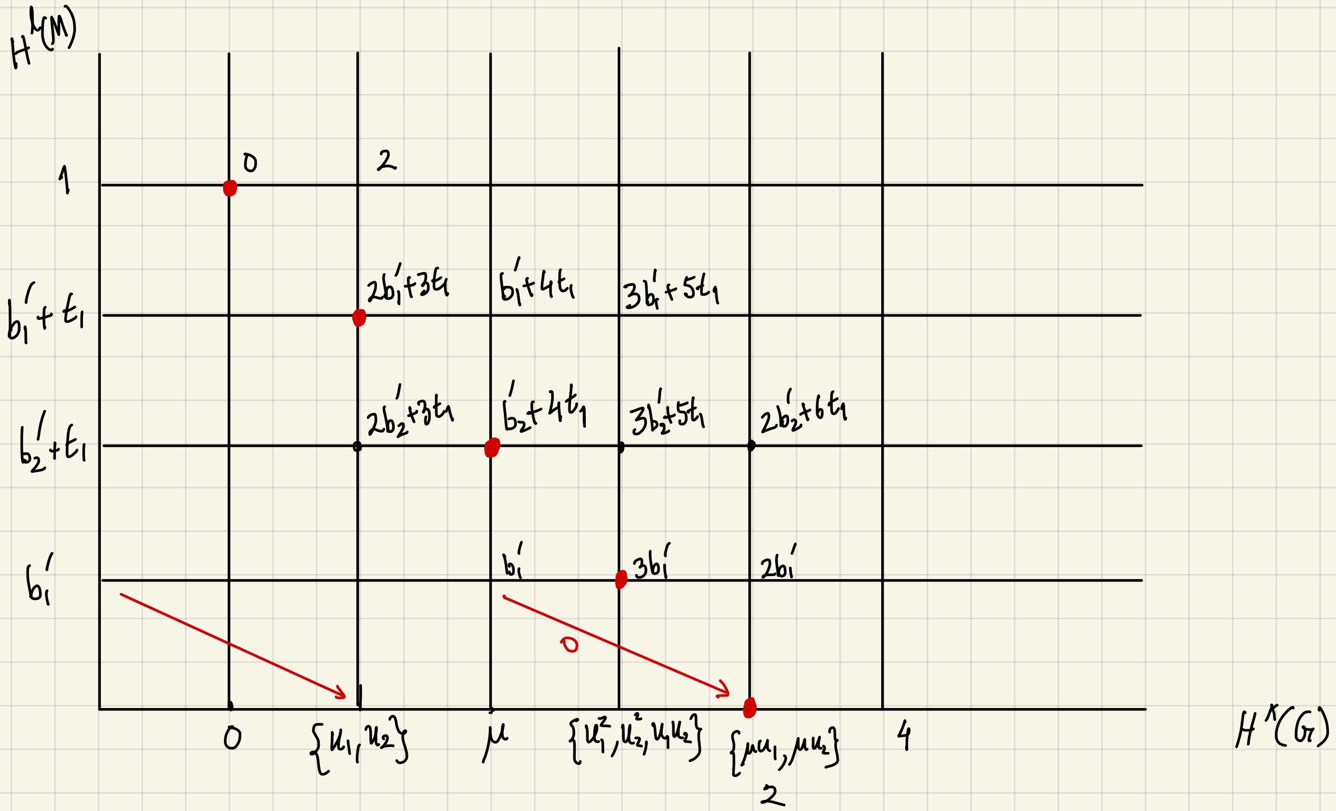
$$\dim E_\infty^{q-k, k} = 0, \quad \text{for } q \geq 5 \text{ odd.}$$

Proof: $\dim H^q_G(M) \approx \sum \dim E_\infty^{q-k, k}$; $q \geq 5$, $k = 0, 1, \dots, 4$.

$(q \text{ odd}) = 0 \Rightarrow E_\infty^{q-k, k} = 0$, $q \geq 5$ odd.

We wanted to make use of Lemma 3, especially the 5-line
$$H^5_G(M) = 0 = E_\infty^{5,5-k}.$$

Dimension Calculations of BSS of action of $\overset{G}{\overset{\parallel}{\mathbb{Z}_2 \times \mathbb{Z}_2}}$ on M^4 :



$$E_2^{3,2} = H^3(G; H^2(M))$$

$$= \underbrace{H^3(G) \otimes H^2(M)}_{\dim\ b_2'+t_1} \oplus \underbrace{Tor(H^4(G), H^2(M))}_{\dim\ 3t_1}.$$

dim $b_2'+4t_1$

$$\left[\ H^*(G) = \mathbb{Z}[u_1, u_2](\mu) \Big/ \left(2u_1, 2u_2, 2\mu, \mu^2 = u_1u_2(u_1+u_2)\right) \qquad |u_i| = 2,\ |\mu| = 3.\ \right]$$

$$\dim E_2^{3,2} = \dim(H^3(G) \otimes H^2(M)) \oplus \dim Tor.$$

$$\dim_{\mathbb{F}_2} H^3(G) \otimes H^2(M) = \dim\left[\mathbb{Z}_2 \otimes \left(\mathbb{Z}^{b_2'} \oplus Tors\, H_1(M)\right)\right]$$
$$= \dim \mathbb{Z}_2^{b_2'} \oplus_+ \dim\left[\mathbb{Z}_2 \otimes Tors\, H_1(M).\right]$$

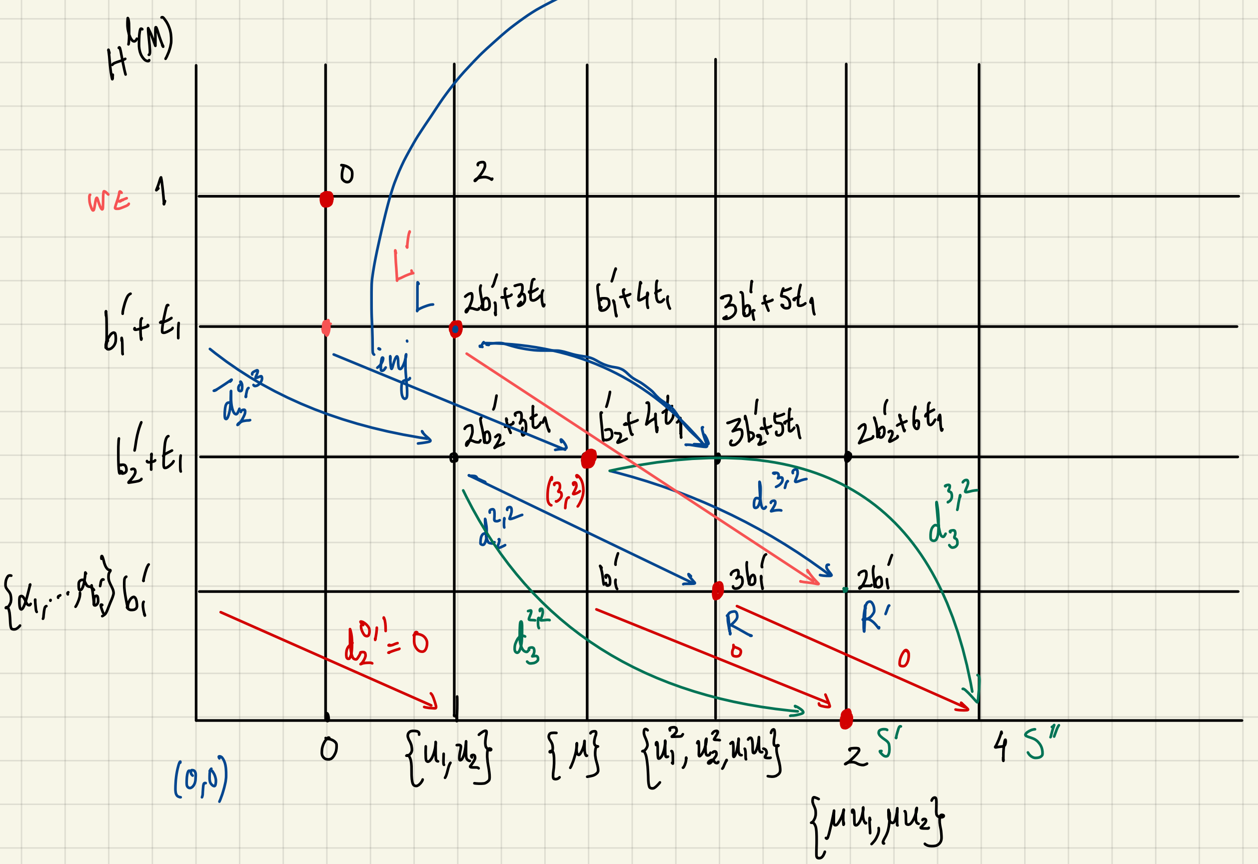
$$= b_2' + t_1$$

$$\dim \mathrm{Tor}\left((\mathbb{Z}_2)^3, \mathbb{Z}^{b_2'} \oplus \mathrm{Tors} H_1(M) \right) = 0 + 3 \underbrace{\dim \mathrm{Tor}(\mathbb{Z}_2, \mathrm{Tors}\, H_1(M))}_{t_1}$$

$$= 3t_1 .$$

$d_2^{1,3} : E_2^{1,3} \longrightarrow E_2^{3,2}$ is injective.

$E_3^{1,3} = \ker d_3^{1,3}$.



$H^0(G; H^2(M)) \simeq H^2(M)$

R , R' → coker.

L ker $d_2^{2,3}$

Goal is to "some how" find $\dim E_\infty^{k,5-k}$ in terms of $b_2', b_1', t_1 \dots$ and use Lemma 3 to get some equalities / inequalities !

Can we get some information on differentials at this stage ?

(i) $\mathrm{Img}\, d_r^{k,r-1} \subseteq \mathrm{Ess}^*(G) \rightarrow$ Essential Cohomology of $G := \mathbb{Z}_p \times \mathbb{Z}_p$
$p \geq 2$ prime.

[G = finite group. $x \in H^n(G)$ is essential element if $x \in \bigcap_{e \neq K \lneq G} \ker \mathrm{Res}_K^G$]

$K \hookrightarrow G$
$BK \to BG$
$\mathrm{Res}_K^G : H^*(G) \to H^*(K)$

If $G = \mathbb{Z}_2 \times \mathbb{Z}_2$; $K < G$.

$H^*(K) = \mathbb{Z}(u)/(2u)$

$|u| = 2.$

Res^G_K :

$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\langle a \rangle$	$\langle b \rangle$	$\langle ab \rangle$
u_1	u	0	u
u_2	0	u	u
μ	0	0	0

$$Res^G_K : H^3(G) \longrightarrow H^3(K) = 0$$

$$\mu \longmapsto 0$$

$u_1, u_2 \notin Ess(G)$; $\mu \in Ess(G)$.

- $d_2^{k,1} = 0$

$$d_2^{k,1} : H^k(G, H^1(M)) \longrightarrow H^{k+2}(G)$$

$$d_2^{k,1}\Big(H^k(G) \otimes \underbrace{H^1(M)}_{\{\alpha_i\}}\Big) = \bigoplus H^k(G)\, \underbrace{d_2^{0,1}(\alpha_i)}_{0}$$

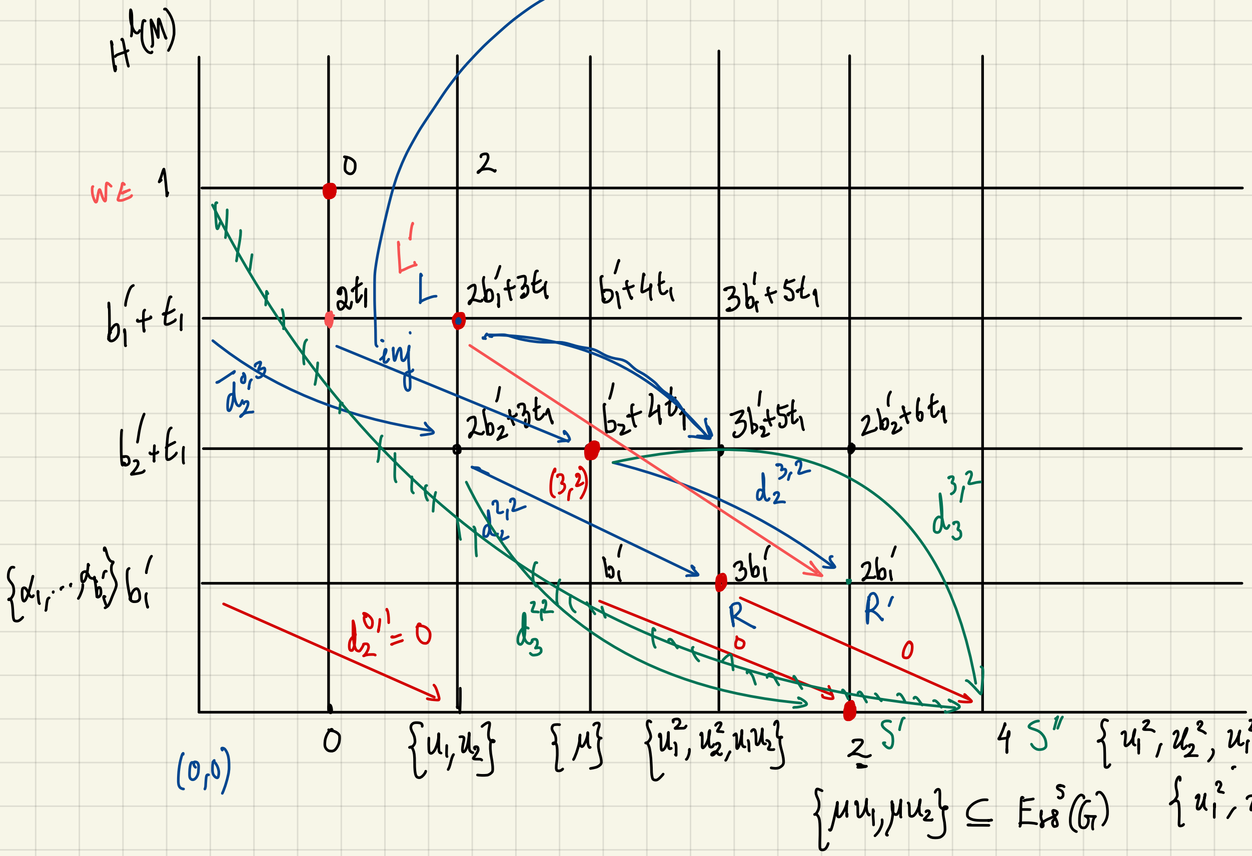
$$= 0.$$

$d_2^{0,4} w \neq 0$; $2w \in \ker d_2^{0,4} = E_3^{0,4}$

① $d_2(w) = 0 = d_3(w) = d_4(w)$, $d_5(w) \neq 0$

$d_2^{1,3}: E_2^{1,3} \longrightarrow E_2^{3,2}$ is injective.

$E_3^{1,3} = \ker d_3^{1,3}$.



$R, R' \to$ coke

$L \quad \ker d_2^{2,3}$

$[\mu^2 = u_1u_2(u_1+u_2)$

Page:	$(5,0)$	$(4,1)$	$(3,2)$	$(2,3)$	$(1,4)$
E_2	2	$3b_1'$	$b_2'+4t_1$	$2b_1'+3t_1$	0
E_3	S'	R	$b_2'+4t_1-2b_1'+R'-2t_1$ $= b_2'-2b_1'+2t_1+R'$	L	0
E_4	S'	R	$b_2'-2b_1'+2t_1+R'-4+S''$	L'	0
$E_\infty = E_5$	$S'-1$	R	″	L''	0

$\dim E_\infty^{5,5-k} = 0$, for each $k = 0, \dots, 4$.

$S' = 1$, $R = 0$, $b_2' - 2b_1' + 2t_1 + R' + S'' - 4 = 0$, $L'' = 0$.

$s'' \geq 3. \longrightarrow$ $b_2' - 2b_1' + 2t_1 + R' \leq 1.$

$t_1 \geq 0$, $R' \geq 0$

$p = 2, \quad H_G^q(M) = \frac{3}{2}\chi(M) \quad , q \geq 6$ even

$$= \frac{3}{2}\left(b_2' - 2b_1' + 2\right)$$

$2 \mid \chi(M) \quad \Rightarrow \quad b_2' - 2b_1'$ is even.

$$\left[b_2' - 2b_1' = 0 \quad ; \quad t_1 = 0 \quad ; \quad R' \leq 1. \right]$$

$\downarrow$ not possible. ✓

$\Rightarrow$

(i) $\left[d_2(w) = 0; \quad d_3(w) \neq 0 \quad d_{r\geq 4}(2w) = 0. \right]$

$$b_2' = 2; \qquad b_1' = 0.$$

[A] $G \curvearrowright M$ Then $b_1'(M) = 0$ and if $b_2' \geq 3$ the G must cyclic and acts semifreely.

[B] $b_2 \leq 2$

(i) $b_2 = 2$, and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$; $M \longrightarrow$ 2-loc. $S^2 \times S^2$

$G = \mathbb{Z}_q \rtimes \mathbb{Z}_2$, q odd ... $\longrightarrow$ q-local $S^2 \times S^2$

(ii) $b_2 = 1$ $G = \mathbb{Z}_3 \rtimes \mathbb{Z}_3$, $\mathbb{Z}_q \rtimes \mathbb{Z}_3$ ... $\mathbb{C}P^2$