Homotopy connectivity of Čech complexes of spheres

Sucharita Mallick

University of Florida

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Motivation

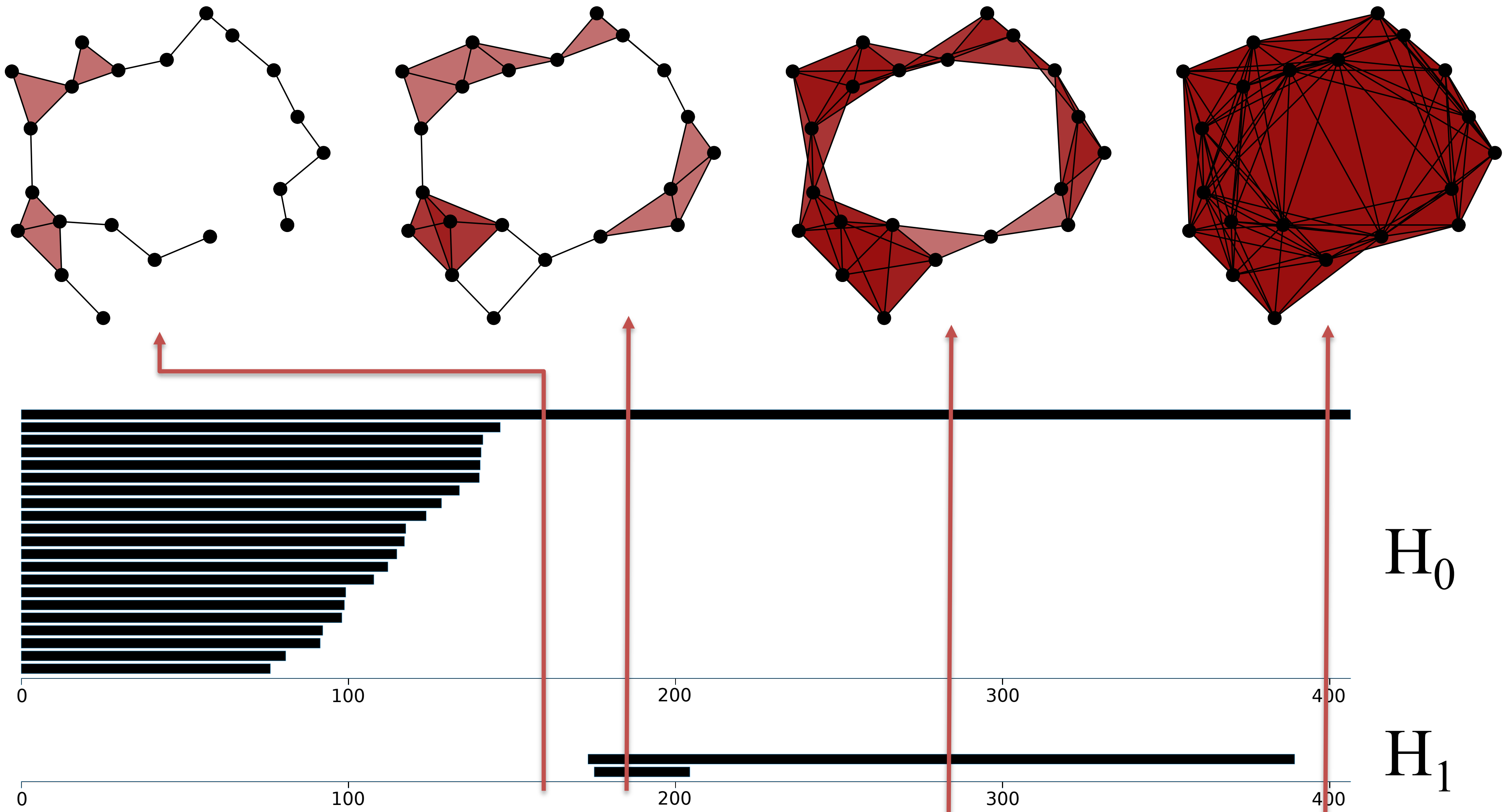
In topological data analysis (TDA), one of the main goals is to estimate the shape of the data. Given a point cloud, one often replaces the points with balls of radius r centered at that point. In persistent homology, the scale r is varied in a range to get a multi-resolution summary of the shape of the data.

Čech complexes are combinatorial representation of the union of the balls. We have vertices for each data point, when two balls intersect there is an edge, when three balls intersect there is a triangle, and so on.

Demonstration of a Čech complex

https://sauln.github.io/blog/nerve-playground/
Demonstration by Nathaniel Saul.

Persistent homology



- Input: Increasing spaces. Output: barcode.
- Significant features persist.
- Cubic computation time in the number of simplices.

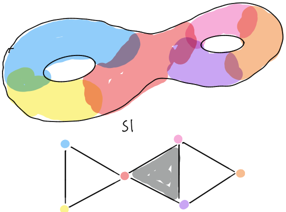
Nerve complex and nerve lemma

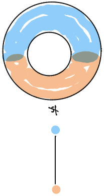
Let $\mathcal{U}$ be a collection of open sets in a topological space. The nerve complex $N(\mathcal{U})$ has

- a vertex for each set in $\mathcal{U}$
- a k-simplex when $(k+1)$ sets intersect.

Nerve lemma

Let $\mathcal{U}$ be a good collection of open sets, then $N(\mathcal{U}) \simeq$ union of sets in $\mathcal{U}$.





Čech complexes

Definition

Let (X, d) be a metric space. For a scale $r > 0$, the *intrinsic* Čech complex $\check{C}(X; r)$ is the nerve of the collection of open balls $\{B_X(x; r)\}_{x \in X}$.

Čech complexes for balls in Euclidean space are well understood because the balls are convex and conditions of the nerve lemma are satisfied.

For a Riemannian manifold M, when the scale r is greater than the convexity radius, the nerve lemma no longer applies. What are the homotopy types of intrinsic Čech complex $\check{C}(M; r)$?

We focus on the case $M = S^n$ and investigate the homotopy types of the intrinsic Čech complex, $\check{C}(S^n; r)$.

Notation

Let S^n be the n-sphere with the geodesic metric and diameter π.

Definition

The intrinsic Čech complex of S^n at scale $r \in (0, \pi)$, $\check{\mathrm{C}}(S^n; r)$ is the nerve of the open balls $\{B_{S^n}(x; r)\}_{x \in S^n}$.

More formally,

$$\check{\mathrm{C}}(S^n; r) = \left\{ \text{finite } \sigma \subset S^n \mid \bigcap_{x \in \sigma} B_{S^n}(x; r) \neq \emptyset \right\}.$$

Definition

For a space X, the homotopy connectivity $\mathrm{conn}(X)$ is the largest integer k such that $\pi_i(X) = 0$ for $i \leq k$.

Related work

- By the nerve lemma, when $0 < r \leq \frac{\pi}{2}$, $\check{\mathrm{C}}(S^n; r) \simeq S^n$.
- [Adamaszek-Adams, 2017] $\check{\mathrm{C}}(S^1; r) \simeq S^{2k+1}$ for $\frac{\pi k}{k+1} < r \leq \frac{\pi(k+1)}{k+2}$.
- [Virk, 2020] For $n \geq 2$, $\check{\mathrm{C}}(S^n; r)$ is simply connected.

Question

For $\delta \in (0, \frac{\pi}{2})$, how does $\mathrm{conn}(\check{\mathrm{C}}(S^n; \pi - \delta))$ change depending on δ?

Main result

We control the connectivity of $\mathrm{conn}(\check{C}(S^n; \pi - \delta))$ in terms of coverings of spheres.

Definition

For a metric space (X, d), the *k-th covering radius* is defined by

$$\mathrm{cov}_X(k) := \inf \left\{ r \geq 0 \mid \exists\ x_1, \ldots, x_k \in X \text{ s.t. } \bigcup_{i=1}^{k} B[x_i; r] = X \right\}.$$

Main theorem

For $n \geq 1$ and $\delta \in (0, \pi)$, if $\mathrm{conn}(\check{C}(S^n; \pi - \delta)) = k - 1$, then $\mathrm{cov}_{S^n}(2k + 2) \leq \delta \leq 2 \cdot \mathrm{cov}_{S^n}(k + 1)$.

Corollary

Definition

For $\delta > 0$, the δ-covering number of a metric space (X, d) is defined by

$$\mathrm{numCover}_X(\delta) := \min \left\{ k \geq 1 \mid \exists\, x_1, \ldots, x_k \in X \text{ s.t. } \bigcup_{i=1}^{k} B[x_i; \delta] = X \right\}.$$

Corollary

For $n \geq 1$, the homotopy type of the Čech complex $\check{\mathrm{C}}(S^n; \pi - \delta)$ changes infinitely many times as δ varies over the range $(0, \pi)$.

Proof: The main theorem can be restated as,

$$\tfrac{1}{2}\mathrm{numCover}_{S^n}(\delta) - 2 \leq \mathrm{conn}(\check{\mathrm{C}}(S^n; \pi - \delta)) \leq \mathrm{numCover}_{S^n}(\tfrac{\delta}{2}) - 2.$$

$\mathrm{numCover}_{S^n}(\delta) \to \infty$ as $\delta \to 0$ but for any $\delta > 0$, $\mathrm{numCover}_{S^n}(\frac{\delta}{2}) < \infty$.

How good are the bounds?

In the case $n = 1$, if $\mathrm{conn}(\check{C}(S^1; \pi - \delta)) = k - 1$ then the upper bound on δ is sharp but the lower bund is off by approximately a factor of 4.

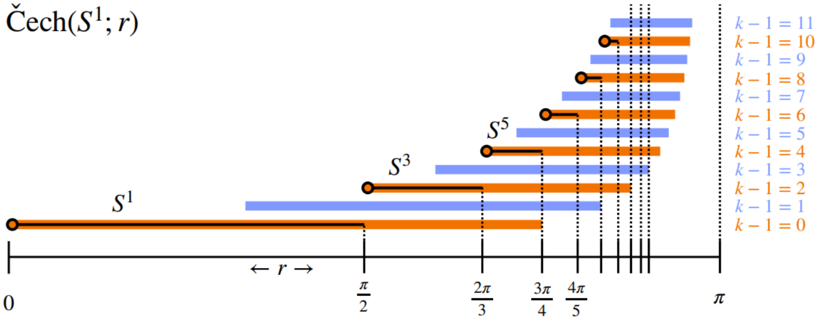


FIGURE 2. The homotopy types of $\check{C}(S^1; r)$ as r varies [1] are indicated by black bars. Theorem 1.1 gives intervals where $\check{C}(S^1; r)$ may have connectivity $k - 1$, which are indicated by colored bars. Left endpoints of orange bars (when $k - 1$ is even) are tight.

Upper bound

Definition

For $0 < \delta < \pi$, the Borsuk graph $\mathrm{Bor}(S^n; \delta)$ has

- the vertex set S^n, the n-sphere, and
- an edge (u, v) when $d(u, v) > \delta$.

Main idea

$\check{\mathrm{C}}(S^n; \pi - \delta)$ is $N(\mathrm{Bor}(S^n; \delta))$, the neighborhood complex of $\mathrm{Bor}(S^n; \delta)$.

Lemma [Lovász, 1978]

For any graph G, the chromatic number satisfies
$\chi(G) \geq \mathrm{conn}(N(G)) + 3$.

Thus we have an upper bound, $\mathrm{conn}(\check{\mathrm{C}}(S^n; \pi - \delta)) \leq \chi(\mathrm{Bor}(S^n; \delta)) - 3$.

Proof of the upper bound

Theorem 1

If $\delta > 2 \cdot \mathrm{cov}_{S^n}(k+1)$ for some $k \geq 1$, then $\mathrm{conn}(\check{\mathrm{C}}(S^n; \pi - \delta)) \leq k - 2$.

Proof.

When $\delta > 2 \cdot \mathrm{cov}_{S^n}(k+1)$ for some $k+1 \geq 2$, then $\chi(\mathrm{Bor}(S^n;\delta)) \leq k+1$ [Moy, 2024]. Using Lovász' bound we conclude that

$$\mathrm{conn}(\check{\mathrm{C}}(S^n; \pi - \delta)) \leq \underbrace{\chi(\mathrm{Bor}(S^n;\delta))}_{\lceil \frac{2\pi}{\delta} \rceil \text{ when } n=1} - 3 \leq k + 1 - 3 = k - 2.$$

□

Lower bound

Theorem 2

If $0 < \delta < \operatorname{cov}_{S^n}(2k+2)$, then $\operatorname{conn}(\check{\mathrm{C}}(S^n; \pi - \delta)) \geq k$.

Corollary [Barmak, 2023]

Let K be a simplicial complex. If any $(2k+2)$ vertices are contained in a common simplex then K is k-connected.

Assumption: The collection of closed star of vertices $\{\mathrm{st}_K(v)\}_{v \in K}$ is locally finite. Where

$$\mathrm{st}_K(v) := \{\sigma \in K \mid \sigma \cup \{v\} \in K\}.$$

Caveat: $\check{\mathrm{C}}(S^n; \pi - \delta)$ does not satisfy this assumption. But we can choose an ε-dense subset X of S^n and work with that.

Proof of the lower bound

Proof.

Choose $\varepsilon > 0$ such that $\delta + \varepsilon < \mathrm{cov}_{S^n}(2k+2)$ and let X be an ε-dense subset of S^n. Let $x_1, \cdots, x_{2k+2}$ be any $(2k+2)$ points in X. As $\delta + \varepsilon < \mathrm{cov}_{S^n}(2k+2)$,

$$\begin{aligned} &\text{No } (2k+2) \text{ closed } (\delta+\varepsilon)\text{-balls cover } S^n \\ \implies &\text{Any } (2k+2) \text{ open } (\pi-\delta-\varepsilon)\text{-balls intersect in } S^n \\ \implies &\text{Any } (2k+2) \text{ open } (\pi-\delta)\text{-balls intersect in } X \\ \implies &[x_1, \cdots, x_{2k+2}] \in \check{\mathrm{C}}(X; \pi-\delta) \\ \implies &\mathrm{conn}(\check{\mathrm{C}}(X; \pi-\delta)) \geq k. \end{aligned}$$

For any $i \leq k$ and $f\colon S^i \to \check{\mathrm{C}}(S^n; \pi-\delta)$, we have $\mathrm{im}(f) \subseteq \check{\mathrm{C}}(X; \pi-\delta)$ for some ε-dense subset X as S^i is compact. Since f is nullhomotopic in $\check{\mathrm{C}}(X; \pi-\delta)$, f is nullhomotopic in $\check{\mathrm{C}}(S^n; \pi-\delta)$. □

Conjectures and open questions

- For $n \geq 1$, is the connectivity of the Čech complex $\check{\mathrm{C}}(S^n; r)$ a non-decreasing function of the scale $r \in (0, \pi)$?
- For $n \geq 1$, does homotopy type of the Čech complex $\check{\mathrm{C}}(S^n; r)$ change only countably many times as r varies over the range $(0, \pi)$? Can Morse theory be used to prove this?
- Is the $\check{\mathrm{C}}(S^n; r)$ homotopy equivalent to a finite CW complex?
- What assumptions are needed on a compact Riemannian manifold M so that $\check{\mathrm{C}}(M; r)$ is homotopy equivalent to a finite-dimensional CW complex?

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