Algebraic Invariants of knots and links

Sujoy Mukherjee

based on joint work with
Louis H. Kauffman, Józef H. Przytycki and Petr Vojtěchovský

Indian Institute of Science Education and Research, Kolkata
DMS Topology and Geometry Seminar
February 25, 2026

Contents

What is a link?

A **link** of m components is a subset of $\mathbb{S}^3$ (or, of $\mathbb{R}^3$) that consists of m disjoint, piecewise linear, simple closed curves. A link of one component is a **knot**.

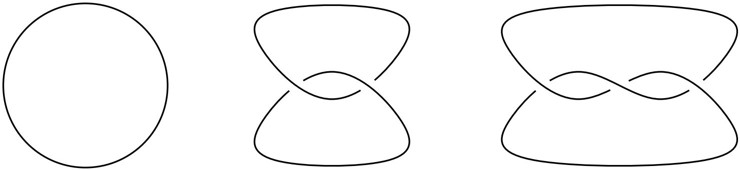

Figure 1: The unknot, the Hopf link and the trefoil knot.

Given a link diagram, what are all the other possible diagrams that represent the same link type?

This question was answered by Reidemeister in the 1920s.

Reidemeister moves

Reidemeister proved that two link diagrams represent the same link if and only if one can be changed into the other by isotopy and a finite sequence of local moves (now known as **Reidemeister moves**) of the three types shown below.

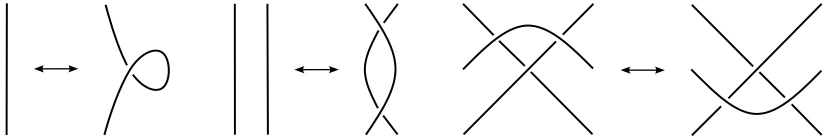

Figure 2: Reidemeister moves 1, 2 and 3.

From Reidemeister moves to self-distributive magmas

Quandles are algebraic structures whose axioms are algebraic translations of the three Reidemeister moves. To see this, we color the arcs of an oriented link diagram with elements of such magmas.

From Reidemeister moves to self-distributive magmas

Quandles are algebraic structures whose axioms are algebraic translations of the three Reidemeister moves. To see this, we color the arcs of an oriented link diagram with elements of such magmas.

Let X be a non-empty set with two binary operations $*$ and $/$ defined on it.

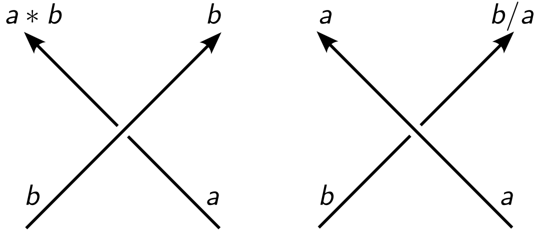

Figure 3: Rules for coloring the arcs of a link diagram with elements of X.

From Reidemeister moves to self-distributive magmas

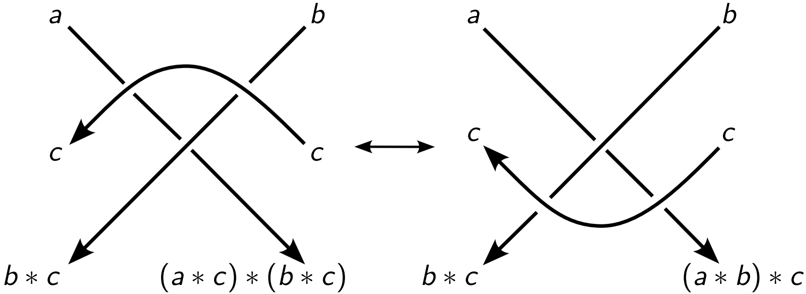

Figure 4: Effect of the third Reidemeister move on coloring.

From Reidemeister moves to self-distributive magmas

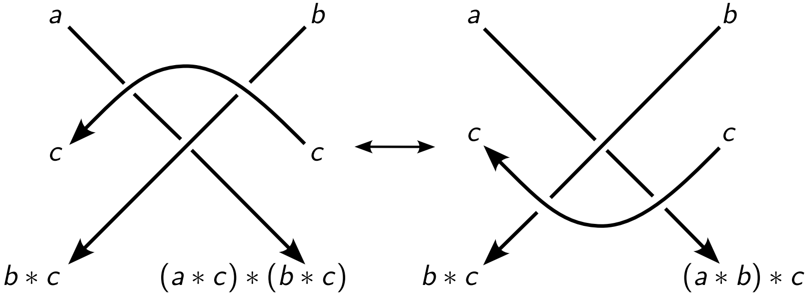

Figure 4: Effect of the third Reidemeister move on coloring.

Therefore, for $a, b, c \in X$, we must have

$$(a * b) * c = (a * c) * (b * c).$$

The other two Reidemeister moves give two more conditions.

From Reidemeister moves to self-distributive magmas

A **shelf** or a self-distributive algebraic structure is a magma $(X, *)$ such that for all $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

A **rack** is a shelf if for all $a, b \in X$, $(a/b) * b = a = (a * b)/b$.

A **quandle** is an idempotent rack, that is, $a * a = a$, for all $a \in X$.

A **spindle** is an idempotent shelf.

Examples

Let $(X, *_0)$ be a magma and $a *_0 b = a$ for all $a, b \in X$. Then $(X, *_0)$ is a quandle called the **trivial quandle**.

Let $(G, +)$ be an Abelian group. Define for $a, b \in G$, $a * b = 2b - a$. Then $(G, *)$ is a quandle called the **Takasaki quandle**. When $G = \mathbb{Z}_n$, $(G, *)$ is also called the **dihedral quandle** and is denoted by R_n.

Let $(G, \cdot)$ be a group. For all $a, b \in G$, let $a * b = b^{-1} \cdot a \cdot b$. Then $(G, *)$ is a quandle called the **conjugation quandle**.

$f_{i,j}$ quandles

Let $\{X_i\}_{i\in\Lambda}$ be non-empty sets and $X = \bigsqcup_{i\in\Lambda} X_i$. For each $i \in \Lambda$, let $f_{i,j} : X_i \longrightarrow X_i$ be a family of functions indexed by $j \in \Lambda$. Further, let $* : X \times X \longrightarrow X$ be defined as follows. For $x_i \in X_i$ and $x_j \in X_j$, $x_i * x_j = f_{i,j}(x_i)$.

1. $(X, *)$ is a shelf if and only if $f_{i,j}f_{i,k} = f_{i,k}f_{i,j}$ for all $i, j, k \in \Lambda$. In other words, for fixed i, the functions $f_{i,j}$ commute.
2. $(X, *)$ is a rack if, in addition to being a shelf, all the functions $f_{i,j}$ for all $i, j \in \Lambda$ are bijections.
3. For the idempotency axiom, we need $f_{i,i} = Id_{X_i}$ for all $i \in \Lambda$.

Examples of finite $f_{i,j}$ quandles

Table 1: $GQ(3 \mid 3)$ with two orbits and $GQ(2 \mid 1 \mid 3)$ with three orbits.

$*$	0	1	2	3	4	5
0	0	0	0	1	1	1
1	1	1	1	2	2	2
2	2	2	2	0	0	0
3	4	4	4	3	3	3
4	5	5	5	4	4	4
5	3	3	3	5	5	5

$*$	0	1	2	3	4	5
0	0	0	0	1	1	1
1	1	1	1	0	0	0
2	2	2	2	2	2	2
3	4	4	4	3	3	3
4	5	5	5	4	4	4
5	3	3	3	5	5	5

The monoid of binary operations

Let X be a set and $* : X \times X \longrightarrow X$ be a binary operation. Let $Bin(X)$ be the collection of all binary operations on X.

Proposition 1.1

$Bin(X)$ has a monoidal structure with composition $*_1 *_2$ given by $a(*_1 *_2)b = (a *_1 b) *_2 b$ and the identity element, denoted by $*_0$, given by $a *_0 b = a$ for all $a, b \in X$.

Remark 1.2

All the invertible elements of $Bin(X)$ form a group denoted by $Bin_{inv}(X)$. If $* \in Bin_{inv}(X)$, then $*^{-1}$ is denoted by $\bar{*}$.

$Bin(X)$

Table 2: The multiplication table of $Bin(X)$ when $|X| = 2$.

	0	1	2	**3**	4	5	**6**	7	8	**9**	10	11	**12**	13	14	15
0	0	0	0	0	5	5	5	5	10	10	10	10	15	15	15	15
1	0	1	0	1	4	5	4	5	10	11	10	11	14	15	14	15
2	0	0	2	2	5	5	7	7	8	8	10	10	13	13	15	15
3	0	1	2	**3**	4	5	**6**	7	8	**9**	10	11	**12**	13	14	15
4	0	4	0	4	1	5	1	5	10	14	10	14	11	15	11	15
5	0	5	0	5	0	5	0	5	10	15	10	15	10	15	10	15
6	0	4	2	**6**	1	5	**3**	7	8	**12**	10	14	**9**	13	11	15
7	0	5	2	7	0	5	2	7	8	13	10	15	8	13	10	15
8	0	0	8	8	5	5	13	13	2	2	10	10	7	7	15	15
9	0	1	8	**9**	4	5	**12**	13	2	**3**	10	11	**6**	7	14	15
10	0	0	10	10	5	5	15	15	0	0	10	10	5	5	15	15
11	0	1	10	11	4	5	14	15	0	1	10	11	4	5	14	15
12	0	4	8	**12**	1	5	**9**	13	2	**6**	10	14	**3**	7	11	15
13	0	5	8	13	0	5	8	13	2	7	10	15	2	7	10	15
14	0	4	10	14	1	5	11	15	0	4	10	14	1	5	11	15
15	0	5	10	15	0	5	10	15	0	5	10	15	0	5	10	15

Remark 1.3

When $|X| = 2,\ Bin_{inv}(X) = \{3, 6, 9, 12\} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.$

Multi-shelves and the Mezera embedding

$S \subset Bin(X)$ is called a **distributive set** if, for all $*_\alpha, *_\beta \in S$,

$$(a *_\alpha b) *_\beta c = (a *_\beta c) *_\alpha (b *_\beta c).$$

The pair (X, S) is called a **multi-shelf** when $S \in Bin(X)$ is a distributive set.
We can define **multi-spindles**, **multi-racks**, **multi-quandles**, **multi-associative shelves**, etc. analogously.

Multi-shelves and the Mezera embedding

$S \subset Bin(X)$ is called a **distributive set** if, for all $*_\alpha, *_\beta \in S$,

$$(a *_\alpha b) *_\beta c = (a *_\beta c) *_\alpha (b *_\beta c).$$

The pair (X, S) is called a **multi-shelf** when $S \in Bin(X)$ is a distributive set.
We can define **multi-spindles**, **multi-racks**, **multi-quandles**, **multi-associative shelves**, etc. analogously.

Theorem 1.4 (Mezera, 2015)

Every group G embeds in $Bin(G)$. The embedding, $\phi : G \longrightarrow Bin(G)$, sends g to $*_g$, where $a *_g b = ab^{-1}gb$ (also known as the half-conjugation operation).

Proposition 1.5

$(G, \{*_g\}_{g \in G})$ *is a half-conjugation* **multi-rack**.

Half-conjugation multi-racks

For $G = \mathbb{Z}_n$, the half-conjugation operations correspond to cyclic racks. This follows as, $x *_k y = xy^{-1}ky = x - y + k + y = x + k$, where $k \in \mathbb{Z}_n$.

For example, when $G = \mathbb{Z}_3$, we have the following binary operations:

$*_0$	0	1	2
0	0	0	0
1	1	1	1
2	2	2	2

$*_1$	0	1	2
0	1	1	1
1	2	2	2
2	0	0	0

$*_2$	0	1	2
0	2	2	2
1	0	0	0
2	1	1	1

In general, $(\mathbb{Z}_n, \{*_0, \dots, *_{n-1}\})$ is a half-conjugation multi-rack.

Contents

Pre-simplicial modules to chain complexes

Definition 2.1

A **pre-simplicial module** $(C_n, d_{i,n})$ consists of a sequence of R-modules C_n over a ring R and face maps $d_{i,n} : C_n \longrightarrow C_{n-1}$ for $0 \leq i \leq n$, such that for $i < j$,

$$d_{i,n} \circ d_{j,n+1} = d_{j-1,n} \circ d_{i,n+1}.$$

Pre-simplicial modules to chain complexes

Definition 2.1

A **pre-simplicial module** $(C_n, d_{i,n})$ consists of a sequence of R-modules C_n over a ring R and face maps $d_{i,n} : C_n \longrightarrow C_{n-1}$ for $0 \leq i \leq n$, such that for $i < j$,

$$d_{i,n} \circ d_{j,n+1} = d_{j-1,n} \circ d_{i,n+1}.$$

Proposition 2.2

Let $(C_n, d_{i,n})$ be a pre-simplicial module. Let $\partial_n : C_n \longrightarrow C_{n-1}$ be given by

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_{i,n}.$$

Then, $\partial_n \circ \partial_{n+1} = 0$, so that, (C_n, ∂_n) is a chain complex.

Definition 2.3 (Przytycki, 2010)
Let $C_n^M = \mathbb{Z}X^{n+1}$, where $(X, \{*_1, *_2, \ldots, *_p\})$ is a multi-shelf. Let $a_1, a_2, \ldots, a_p \in \mathbb{Z}$ and $d_{i,n} : C_n^M \longrightarrow C_{n-1}^M$ be given by:

$$d_{i,n}(x_0, \ldots, x_n) = \begin{cases} \sum_{k=1}^{p} a_k\{(x_1, x_2, \ldots, x_n)\} & \text{if } i = 0 \\ \sum_{k=1}^{p} a_k\{(x_0 *_k x_i, \ldots, x_{i-1} *_k x_i, x_{i+1}, \ldots, x_n)\} & \text{if } i \neq 0 \end{cases}.$$

Now, to have the multi-term distributive homology chain complex, let $\partial_n : C_n^M \longrightarrow C_{n-1}^M$ be given by

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_{i,n}.$$

Remark 2.4

In the previous definition,

- *if $p = 1$ and $a_1 = 1$, we get one-term distributive homology.*
- *if $p = 2$, $a_1 = 1$, and $a_2 = -1$, we obtain rack homology.*

Definition 2.5

Let (X, S) be a multi-spindle. The **degenerate chain complex**, $C^D(X)$, is the sub-complex of $C(X)$ generated by tuples, $(x_0, \ldots, x_n)$, with repetitions, i. e. $x_i = x_{i+1}$, for some $0 \leq i < n$. The quotient $C^N(X) = \dfrac{C(X)}{C^D(X)}$ is called the **normalized complex** of X.

The homology of the normalized complex corresponding to rack homology is called **quandle homology**.

Degenerate homology from normalized homology

Theorem 2.6 (Przytycki-Putyra, 2016)

The degenerate multi-term distributive homology of a multi-spindle is determined by the normalized multi-term distributive homology.

Remark 2.7

A lattice L, with the four binary operations: join ($\vee$), *meet* ($\wedge$), *the left-trivial operation* ($\vdash$), *and the right-trivial operation* ($\dashv$), *is a multi-spindle:* $(L, \{\vee, \wedge, \vdash, \dashv\})$.

Przytycki and Putyra have computed the multi-term distributive homology of finite distributive lattices completely.

On rack homology

Theorem 2.8 (Litherland-Nelson, 2003)

For a quandle $(X, *)$, the long exact sequence of quandle homology

$$\longrightarrow H^Q_{n+1}(X) \longrightarrow H^D_n(X) \longrightarrow H^R_n(X) \longrightarrow H^Q_n(X) \longrightarrow H^D_{n-1}(X) \longrightarrow$$

splits into short exact sequences

$$0 \longrightarrow H^D_n(X) \longrightarrow H^R_n(X) \longrightarrow H^Q_n(X) \longrightarrow 0.$$

On rack homology

Theorem 2.8 (Litherland-Nelson, 2003)

For a quandle $(X,*)$, the long exact sequence of quandle homology

$$\longrightarrow H^Q_{n+1}(X) \longrightarrow H^D_n(X) \longrightarrow H^R_n(X) \longrightarrow H^Q_n(X) \longrightarrow H^D_{n-1}(X) \longrightarrow$$

splits into short exact sequences

$$0 \longrightarrow H^D_n(X) \longrightarrow H^R_n(X) \longrightarrow H^Q_n(X) \longrightarrow 0.$$

Theorem 2.9 (Etingof-Grana, 2003)

Let $\mathcal{O}$ be the number of orbits of a finite rack $(X,*)$ with respect to the action of X on itself by right multiplication. Then,

- *$rank(H^R_n(X)) = \mathcal{O}^{n+1}$, and*
- *$rank(H^Q_n(X)) = \mathcal{O}(\mathcal{O}-1)^n$, if additionally $(X,*)$ is a quandle.*

On torsion in rack homology

Theorem 2.10 (Niebrzydowski-Przytycki, 2009)
$H^R_n(X)$ for $n \geq 2$ contains $\mathbb{Z}_p$ torsion, where X is the dihedral quandle of order p.

Theorem 2.11 (Przytycki-Yang, 2015)
Let $(Q, *)$ be a finite quasigroup quandle. Then the torsion subgroup of $H^R_n(Q)$ is annihilated by $|Q|$.

Theorem 2.12 (M.-Przytycki, 2018)
Let $(X, *) = GQ(o_1 \mid o_2)$ be a finite $f_{i,j}$ quandle. Then,

$$H^R_1(X) = \mathbb{Z}^4 \oplus \mathbb{Z}^2_{gcd(o_1, o_2)}.$$

Contents

What is multi-virtual knot theory?

Multi-virtual knot theory was introduced by Kauffman in 2024. It generalizes virtual knot theory by allowing multiple types of virtual crossings.

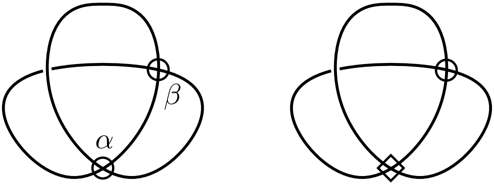

Figure 5: Two conventions for drawing multi-virtual link diagrams.

Let T be a fixed set whose elements will be called **types** and denoted generically by α, β, γ, etc. A **multi-virtual link diagram** (of type T) is a planar drawing of a link in which classical crossings are depicted as usual and where every virtual crossing is assigned a type from T.

Equivalence of multi-virtual link diagrams

Two multi-virtual link diagrams are said to be **equivalent** if one is obtained from the other by finitely many applications of isotopy, classical Reidemeister moves and multi-virtual detour moves.

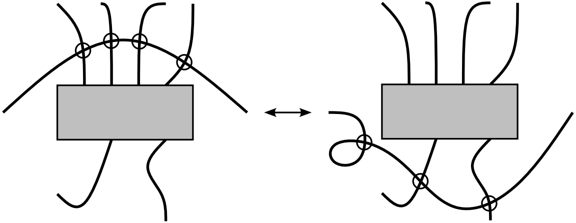

Figure 6: A multi-virtual detour move.

To construct link invariants, a common approach is to check invariance of the 'function' under Reidemeister moves.

Equivalence of multi-virtual link diagrams

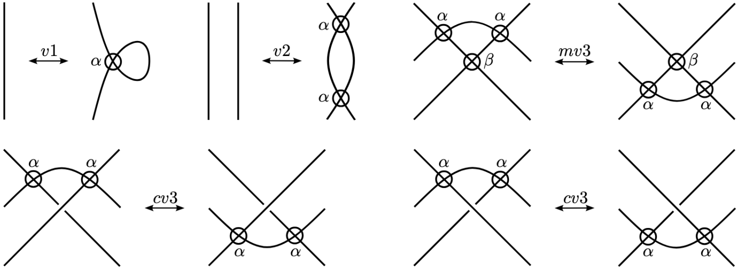

Figure 7: The multi-virtual detour move is equivalent to the moves $v1$, $v2$, $cv3$ and $mv3$.

Theorem 3.1 (Kauffman, M., Vojtěchovský, 202?)

Two multi-virtual link diagrams are equivalent if and only if one is obtained from the other by finitely many applications of isotopy, the classical Reidemeister moves, and the multi-virtual Reidemeister moves $v1$, $v2$, $cv3$ and $mv3$.

Equivalence of multi-virtual link diagrams

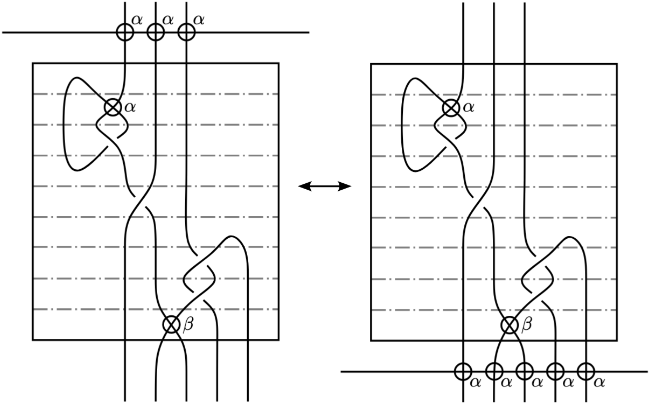

Figure 8: Example of a multi-virtual detour move realized as a sequence of the moves $v1$, $v2$, $cv3$, and $mv3$.

Following Polyak's work, we first obtain a generating set of Reidemeister moves for oriented multi-virtual links.

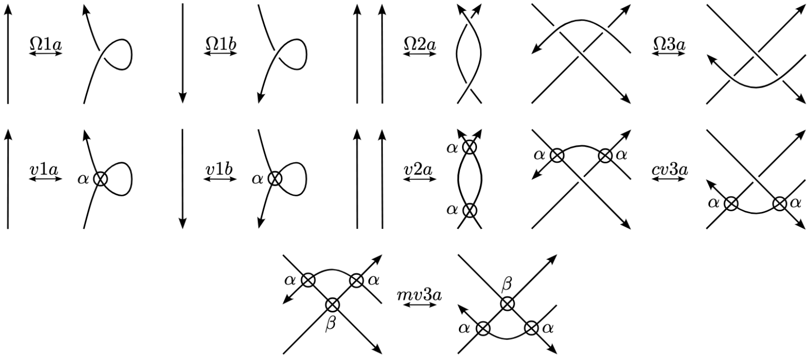

Figure 9: A generating set of Reidemeister moves for oriented multi-virtual links.

Trivial or not?

A central goal in knot theory is to distinguish knots.

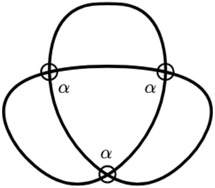

Figure 10: Are these multi-virtual trefoil knots equivalent?

Trivial or not?

A central goal in knot theory is to distinguish knots.

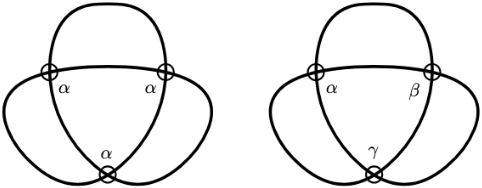

Figure 10: Are these multi-virtual trefoil knots equivalent?

It is known that, there are only finitely many virtual knots with n (fixed) classical crossings. However, there are infinitely many non-equivalent multi-virtual knots with a single classical crossing.

To prove this, we will generalize the well-known coloring invariants for oriented classical links into invariants for oriented multi-virtual links.

Coloring multi-virtual links with operator quandles

Using Manturov's idea of using a quandle automorphism in the coloring rule at a virtual crossing, we construct invariants of multi-virtual links. An **operator quandle** is a quandle $(Q, *)$ together with a list A of its pairwise commuting automorphisms.

We will be assigning automorphisms from the list A to virtual crossings, with all virtual crossings of the same type assigned the same automorphism. For multi-virtual link diagrams of type T, we will denote the list of automorphisms by A_T.

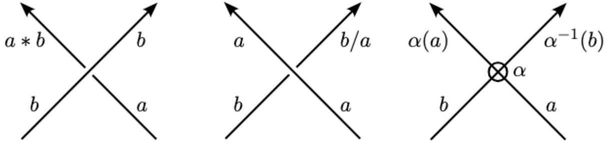

Figure 11: Rules for coloring multi-virtual link diagrams with operator quandles.

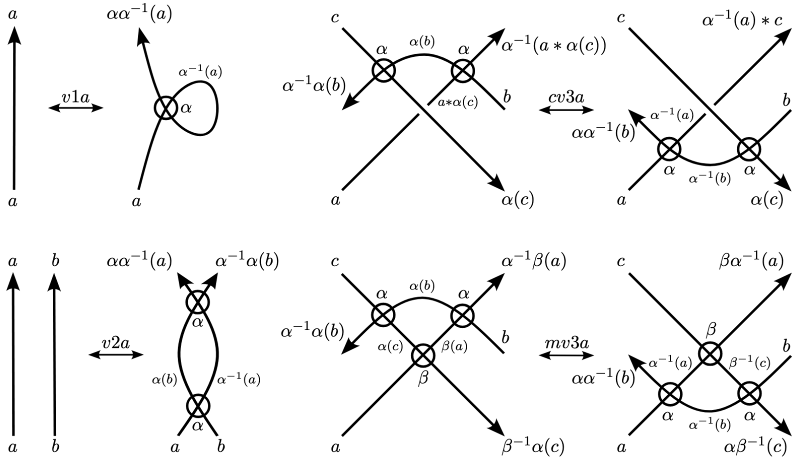

Figure 12: Compatibility of multi-virtual Reidemeister moves with colorings by operator quandles.

The operator quandle coloring invariant

Theorem 3.2 (Kauffman, M., Vojtěchovský, 202?)

Let D_1 and D_2 be two equivalent oriented multi-virtual link diagrams with virtual crossings assigned types from T. Let $(Q,*)$ be an operator quandle and let A_T denote the set of pairwise commuting automorphisms of $(Q,*)$. Then

$$Col(D_1, Q, *, A_T) = Col(D_2, Q, *, A_T).$$

The operator quandle coloring invariant

Theorem 3.2 (Kauffman, M., Vojtěchovský, 202?)

Let D_1 and D_2 be two equivalent oriented multi-virtual link diagrams with virtual crossings assigned types from T. Let $(Q, *)$ be an operator quandle and let A_T denote the set of pairwise commuting automorphisms of $(Q, *)$. Then

$$Col(D_1, Q, *, A_T) = Col(D_2, Q, *, A_T).$$

Next, building on the work done by Carter, Jelsovsky, Kamada, Langford, Saito for classical links and Kazakov for virtual links, we generalize the well-known quandle 2-cocycle invariant to the multi-virtual setting.

2-cocycles of operator quandles

Given a quandle $(Q, *)$ and a group $(G, \cdot)$, a map $\phi : Q \times Q \to G$ is a **quandle 2-cocycle** of $(Q, *)$ with values in G if for all $a, b, c \in Q$,

- $\phi(a, a) = 0$ and
- $\phi(a, b) + \phi(a * c, b * c) = \phi(a, c) + \phi(a * b, c)$.

2-cocycles of operator quandles

Given a quandle $(Q, *)$ and a group $(G, \cdot)$, a map $\phi : Q \times Q \to G$ is a **quandle 2-cocycle** of $(Q, *)$ with values in G if for all $a, b, c \in Q$,

- $\phi(a, a) = 0$ and
- $\phi(a, b) + \phi(a * c, b * c) = \phi(a, c) + \phi(a * b, c)$.

Let D be an oriented multi-virtual link diagram with its virtual crossings assigned types from T. Let $(Q, *)$ be an operator quandle and let A_T denote the set of pairwise commuting automorphisms of $(Q, *)$.

$\phi : Q \times Q \to G$ is an **operator quandle 2-cocycle** of $(Q, *)$ with values in G if it is a quandle 2-cocycle of $(Q, *)$ that satisfies the condition

$$\phi(a, b) = \phi(\alpha(a), \alpha(b))$$

for all $a, b \in Q$ and $\alpha \in A_T$.

The operator quandle 2-cocycle invariant

For a classical crossing c of D and an operator quandle coloring C of D by elements of $(Q, *, A_T)$, let $\phi(a, b)^{\epsilon}$ be the weight assigned to c following the rule shown in the figure below, where $\epsilon = 1$ if c is a positive crossing and $\epsilon = -1$ if c is a negative crossing. Let

$$\mathrm{Coc}(D, Q, *, A_T, \phi) = \sum_{C} \prod_{c} \phi(a, b)^{\epsilon},$$

where the sum is taken over all operator quandle colorings C of D by elements of $(Q, *, A_T)$, and the product is taken over all classical crossings c of D.

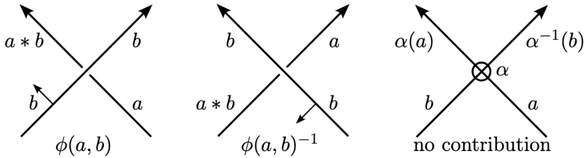

Figure 13: Contributions of crossings to the 2-cocyle invariant.

The operator quandle 2-cocycle invariant

Theorem 3.3 (Kauffman, M., Vojtěchovský, 202?)

Let D_1 and D_2 be two equivalent oriented multi-virtual links with virtual crossings assigned types from T. Let $(Q, *)$ be an operator quandle and let A_T denote the set of pairwise commuting automorphisms of $Q, *)$. Let $(G, \cdot)$ be a group and ϕ an operator quandle 2-cocycle. Then

$$\mathrm{Coc}(D_1, Q, *, A_T, \phi) = \mathrm{Coc}(D_2, Q, *, A_T, \phi).$$

Theorem 3.3 (Kauffman, M., Vojtěchovský, 202?)

Let D_1 and D_2 be two equivalent oriented multi-virtual links with virtual crossings assigned types from T. Let $(Q, *)$ be an operator quandle and let A_T denote the set of pairwise commuting automorphisms of $Q, *)$. Let $(G, \cdot)$ be a group and ϕ an operator quandle 2-cocycle. Then

$$\mathrm{Coc}(D_1, Q, *, A_T, \phi) = \mathrm{Coc}(D_2, Q, *, A_T, \phi).$$

If D has no classical crossings, the empty product $\prod_c \phi(a, b)^\epsilon$ is equal to 1. If D has no operator quandle colorings by $(Q, *, A_T)$, we set the value of $\mathrm{Coc}(D, Q, *, A_T, \phi)$ to be equal to 0.

Permutational equivalence of multi-virtual link diagrams

Let $f : T \to T$ be a bijection and let D be a multi-virtual link diagram. Then $f(D)$ is the diagram obtained from D by renaming all types of virtual crossings according to f. In this context, f will be referred to as a **retyping**.

We say that two multi-virtual link diagrams are **permutationally equivalent** if one is obtained from the other by a finite sequence of retypings, isotopy and multi-virtual Reidemeister moves.

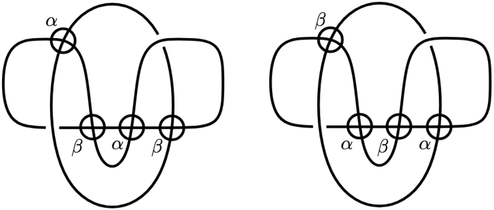

Figure 14: Are these permutationally equivalent links equivalent?

An infinite family of multi-virtual knots

Let $K(n)$ be the oriented multi-virtual knot obtained by a sequence of $2n$ twists with virtual crossings that alternate between distinct types α and β, followed by a single classical crossing, followed by arcs closing the knot.

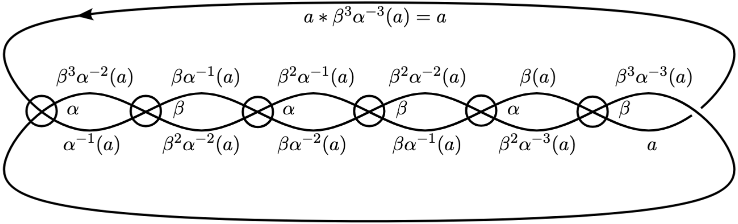

Figure 15: The oriented multi-virtual knot $K(3)$ colored with an operator quandle $(Q, *)$ having automorphisms α and β.

For $p < q$, odd primes, we can show that $K(p) \neq K(q)$ by using the dihedral quandle R_p and its automorphisms $x \mapsto x + 1$ and $x \mapsto x$.

Multi-virtual knots and classical crossing number

Clearly, organizing multi-virtual knots by the number of classical crossings is not a good choice. Using existing tables of virtual knots organized by the number of classical crossings is a good starting point.

Multi-virtual knots and classical crossing number

Clearly, organizing multi-virtual knots by the number of classical crossings is not a good choice. Using existing tables of virtual knots organized by the number of classical crossings is a good starting point.

Let $V = \{2_1, 3_1, 3_2, \ldots, 3_7\}$ be the set of diagrams of all virtual knots with at most 3 classical crossings from Green's catalog. Let M be the set of all oriented multi-virtual knot diagrams with a virtual projection in V.

Multi-virtual knots and classical crossing number

Clearly, organizing multi-virtual knots by the number of classical crossings is not a good choice. Using existing tables of virtual knots organized by the number of classical crossings is a good starting point.

Let $V = \{2_1, 3_1, 3_2, \ldots, 3_7\}$ be the set of diagrams of all virtual knots with at most 3 classical crossings from Green's catalog. Let M be the set of all oriented multi-virtual knot diagrams with a virtual projection in V.

The operator quandle coloring invariant and the operator quandle 2-cocycle invariant can distinguish the multi-virtual links in M completely using suitable operator quandles.

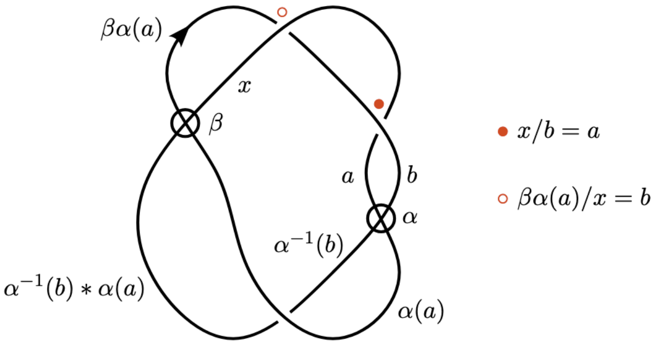

Figure 16: The virtual knot 3_7 and its operator quandle coloring conditions. Here, $x = \beta^{-1}(\alpha^{-1}(b) * \alpha(a))$.

Contents

Future research

Multi-term distributive homology:

- Study the monoid of n-ary operations.
- Study torsion in the multi-term distributive homology of half-conjugation multi-racks and $f_{i,j}$ multi-shelves.
- Construct multi-term homology of Yang-Baxter operators.

Multi-virtual knot theory:

- Start cataloging multi-virtual knots by total crossing number.
- Generalize the chromatic bracket polynomial.
- Use quandle rings to develop coloring invariants for multi-virtual links.
- Generalize invariants such as the Yang-Baxter cocycle invariant for virtual links.

Thanks for listening!