Topological Rigidity of Two- and Three-Dimensional Manifolds

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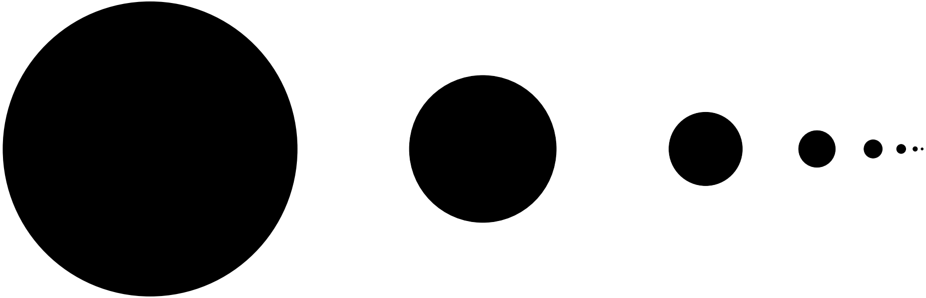
Outline

- Motivation
- Basic Notions
- Homotopy Types vs. Homeomorphism Types
- Hierarchy and Topological Rigidity in Dimensions Two and Three
- Topological Rigidity in Dimension Four and Beyond

Continuously Deformable Spaces and Their Identity



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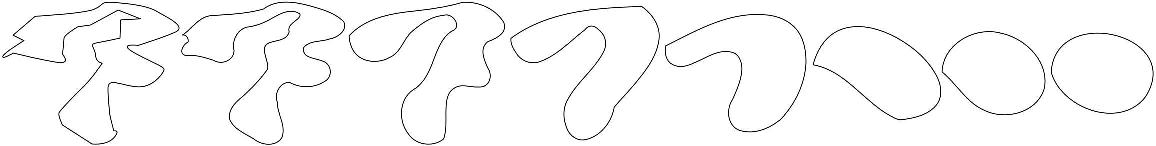


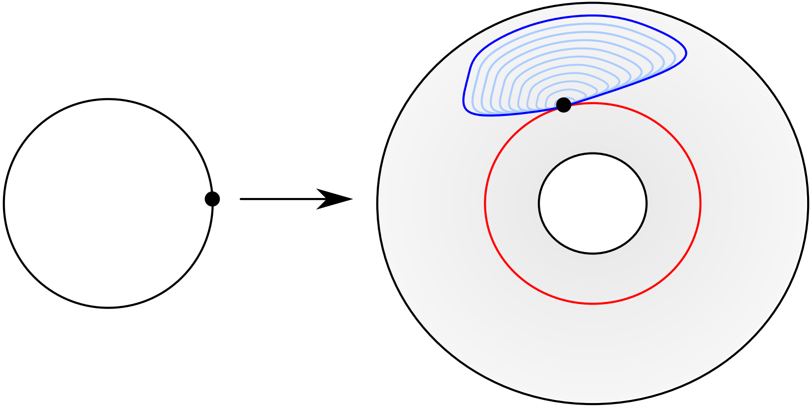
When Spaces Are the Same: Homeomorphisms

Definition. A *homeomorphism* $f\colon X \to Y$ between topological spaces is a (continuous) map such that there exists a (continuous) map $g\colon Y \to X$ with

$$g \circ f = \mathrm{id}_X \quad \text{and} \quad f \circ g = \mathrm{id}_Y .$$

Homotopy: Continuous Deformation of Maps

Definition. Two maps $f, g\colon X \to Y$ are said to be *homotopic relative to* a subset $A \subseteq X$ if there exists a map $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$, and $H(a,t) = f(a) = g(a)$ for all $(a,t) \in A \times [0,1]$.



The Fundamental Group: A Tool for Counting Inequivalent Loops in Spaces

Definition. Let x_0 be a point in the space X. The set of all maps $\ell\colon \mathbb{S}^1 \to X$ with $\ell(1) = x_0$, under the equivalence relation of 'homotopy relative to $\{1\}$', forms a group, denoted by $\pi_1(X, x_0)$, called the *fundamental group* of (X, x_0).

- The multiplication is given by the concatenation of loops.
- The identity element is given by the constant loop.
- The inverse is given by running around the loop in the opposite direction.

Example. $\pi_1(\mathbb{R}^n) = \{1\}$ for all $n \geq 1$, $\pi_1(\mathbb{S}^1) = \mathbb{Z}$, $\pi_1(\mathbb{S}^n) = \{1\}$ for all $n \geq 2$.

When Spaces Are Deformable: Homotopy Equivalences

Definition. A *homotopy equivalence* $f\colon X \to Y$ between topological spaces is a map such that there exists a map $g\colon Y \to X$ with $g \circ f$ homotopic to id_X and $f \circ g$ homotopic to id_Y.

- A homeomorphism is a homotopy equivalence.
- The converse is not true in general. For instance, $\mathbb{R}^n$ and $\mathbb{R}^m$ are homotopy equivalent for all m, n, but they are homeomorphic if and only if $m = n$.

Manifolds: Spaces that Are Locally Euclidean

Definition. An *n dimensional manifold with boundary* is a Hausdorff, second countable space M such that for every point $x \in M$, there exists an open set U containing x and a homeomorphism $U \to \mathbb{R}^n$ or a homeomorphism $U \to [0, \infty) \times \mathbb{R}^{n-1}$.

The *boundary* of the manifold M, denoted ∂M, is the set of points that admit only neighborhoods homeomorphic to $[0, \infty) \times \mathbb{R}^{n-1}$.

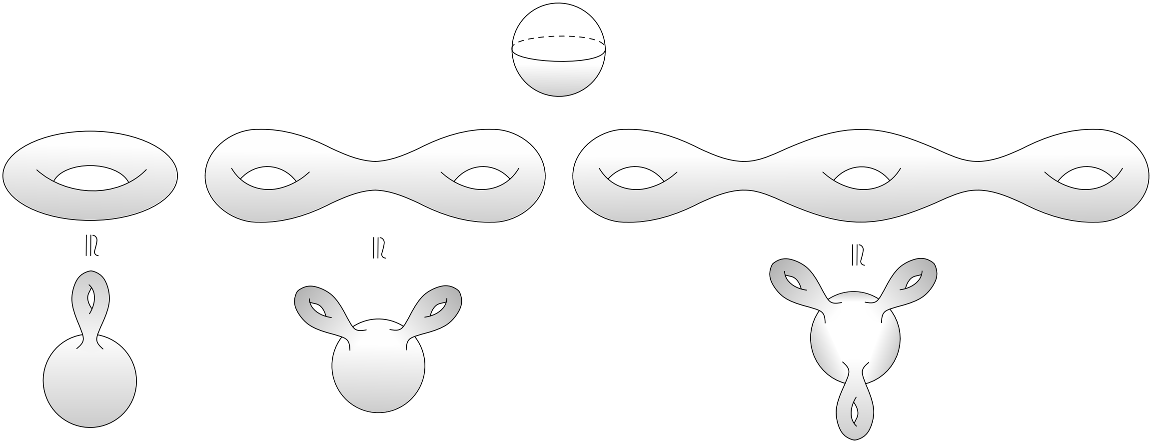
Convention. We will consider manifolds that are path-connected and *orientable* (e.g., π_1 does not have a subgroup of index two).

Classification of 1-Manifolds

Every 1-dimensional manifold is homeomorphic to exactly one of the following:

- $\mathbb{S}^1$
- $(0, 1)$
- $[0, 1)$
- $[0, 1]$

Classification of Compact Boundaryless 2-Manifolds



Examples of Compact 3-Manifolds

- $\mathbb{S}^3$.
- $\mathbb{S}^1 \times \Sigma$, where Σ is a compact surface.
- Handlebody: The compact region of $\mathbb{R}^3$ bounded by a compact, boundaryless surface.
- Knot complement: The space obtained by removing the interior of a tubular neighborhood of a smooth embedding of $\mathbb{S}^1$ into $\mathbb{S}^3$.
- Lens space $L(p, q)$, where $p \neq 0$ and $\gcd(p, q) = 1$: The orbit space of the following "nice" action:

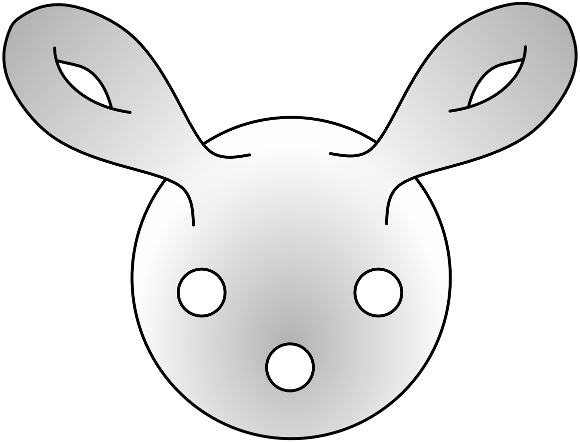
$$\mathbb{Z}_p \times \mathbb{S}^3 \ni \big([k], (z_1, z_2)\big) \longmapsto \left(e^{i\frac{2k\pi}{p}} z_1, e^{i\frac{2k\pi q}{p}} z_2\right) \in \mathbb{S}^3, \quad |z_1|^2 + |z_2|^2 = 1.$$

Homotopy Type vs. Homeomorphism Type

Question. Let M and N be compact n-manifolds with homeomorphic boundaries, i.e., $\partial M \cong \partial N$. Suppose M is homotopy equivalent to N. Is M homeomorphic to N?

Construction of Compact Surfaces

Let g and b be non-negative integers. Suppose $S_{g,b}$ is a compact surface obtained by first attaching g handles to the sphere and then removing the interiors of b pairwise disjoint disks.



Homotopy and Homeomorphism Types of Compact Surfaces

Theorem. Let M be a compact surface. Then M is homeomorphic to $S_{g,b}$ for some $g, b \geq 0$.

Theorem. The following are equivalent:

(1) $S_{g,b}$ is homeomorphic to $S_{h,b'}$.

(2) $\pi_1(S_{g,b})$ is isomorphic to $\pi_1(S_{h,b'})$ and $b = b'$.

(3) $g = h$ and $b = b'$.

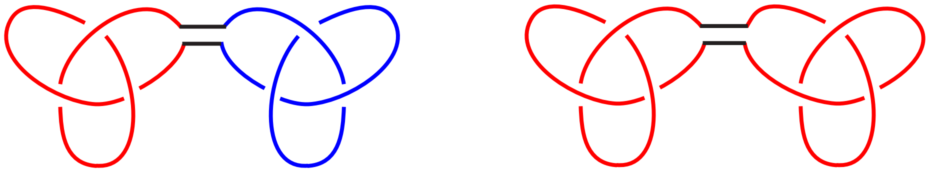
(4) $S_{g,b}$ is homotopy equivalent to $S_{h,b'}$ and $b = b'$.

Homotopy Types vs. Homeomorphism Types in Dimension 3

Theorem. [Reidemeister & Brody] $L(p, q_1) \cong L(p, q_2)$ if and only if $q_1 q_2^{\pm 1} \equiv \pm 1 \pmod p$.

Theorem. [Whitehead] $L(p, q_1) \simeq L(p, q_2)$ if and only if $q_1 q_2^{\pm 1} \equiv \pm t^2 \pmod p$ for some t .

Theorem. [Fox] The complements of the Square Knot and the Granny Knot are homotopy equivalent, non-homeomorphic compact 3-manifolds with homeomorphic boundaries.



Topological Rigidity: Beyond Homotopy Type to Homeomorphism Type

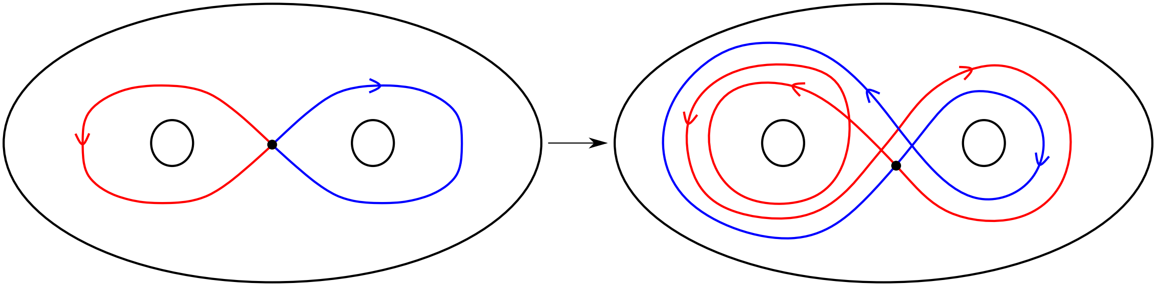
Definition. A compact n-manifold M is *topologically rigid* if every homotopy equivalence $h\colon N \to M$ from a compact n-manifold N that sends ∂N homeomorphically onto ∂M is homotopic to a homeomorphism relative to ∂N.

Borel Uniqueness Conjecture. Every compact aspherical manifold is topologically rigid [Lück, Survey].

Exotic Self Homotopy Equivalences

- There exists a homotopy equivalence $f\colon L(5,1) \to L(5,1)$ sending the generator g of $\pi_1(L(5,1))$ to g^2. Note that f can't be homotopic to a homeomorphism. [Cohen].

- The homotopy equivalence $f\colon S_{0,3} \to S_{0,3}$ that induces a map on $\pi_1(S_{0,3}) = \langle a, b\rangle$ by sending $a \longmapsto a^2 b$ and $b \longmapsto ab$ is not homotopic to a homeomorphism.



Topologically Rigid Compact Manifolds

Theorem. [Nielsen] Every compact 2-manifold is topologically rigid.

Theorem. [Waldhausen] [Gabai-Meyerhoff-N. Thurston] [Turaev] [Perelman] Every compact aspherical 3-manifold (possibly with non-empty boundary) is topologically rigid.

A Large Class of Compact Aspherical 3-Manifolds

Definition. A 3-manifold M is *irreducible* if every smoothly embedded 2-sphere $S \subset M$ bounds a smoothly embedded 3-ball $B \subseteq M$.

Definition. A compact, irreducible 3-manifold M is *Haken* if there exists a two-sided, non-simply connected, compact surface Σ such that $\Sigma \cap \partial M = \partial \Sigma$ and the inclusion induced map $\pi_1(\Sigma) \to \pi_1(M)$ is injective. We call such a Σ an *incompressible surface* in M.

Theorem. Haken manifolds are aspherical.

Theorem. Suppose M is a compact, irreducible 3-manifold such that $H_1(M; \mathbb{Z})$ is infinite. Then M is Haken.

Examples. Knot complements, fiber bundles over $\mathbb{S}^1$ with fiber a closed aspherical surface.

Topological Rigidity of Compact Surfaces with Boundary

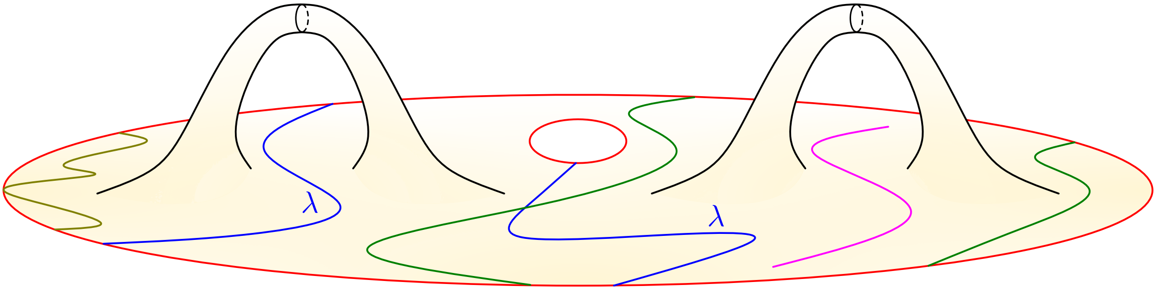
Theorem. [Nielsen] If $f\colon S' \to S$ is a homotopy equivalence between compact surfaces such that $\partial S \neq \varnothing$ and $f|\partial S' \to \partial S$ is a homeomorphism, then f is homotopic to a homeomorphism rel ∂S.

Sketch of proof. The proof will be based on induction on the complexity $C(S) \coloneqq 2g(S) + \sharp\partial S$.

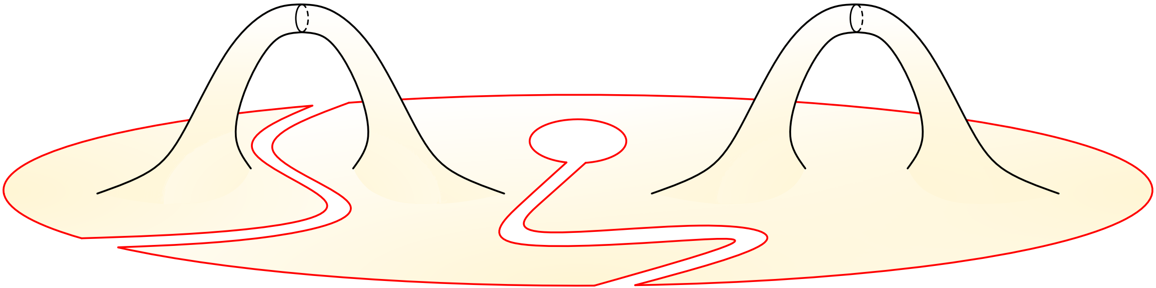
- If S is a disk, then by the Alexander trick, we are done.

- So, from now on, assume that S is not a disk.

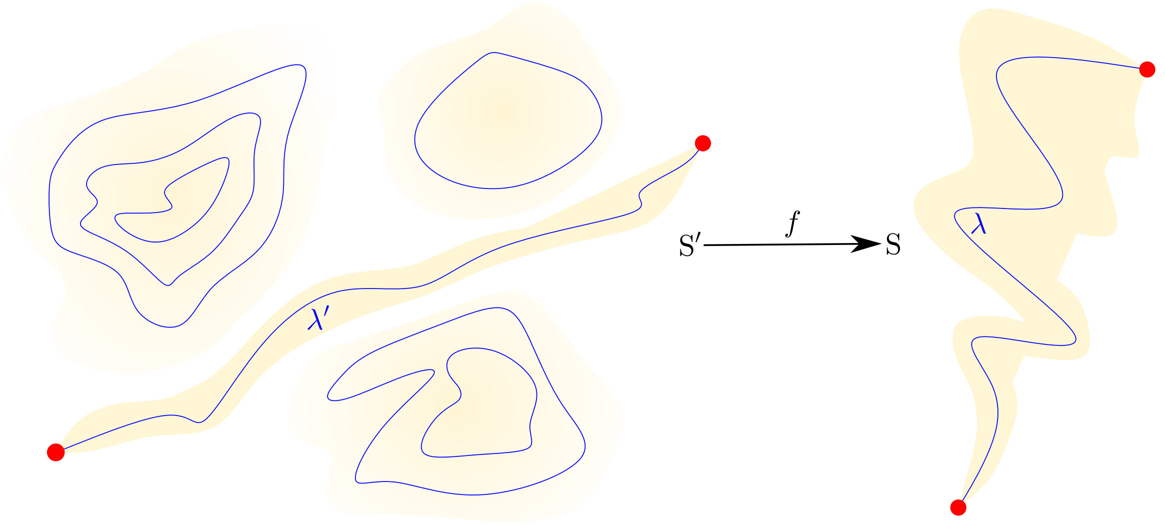
- Pick a non-separating embedded copy λ of $[0, 1]$ in S such that $\lambda \cap \partial S = \partial\lambda$.



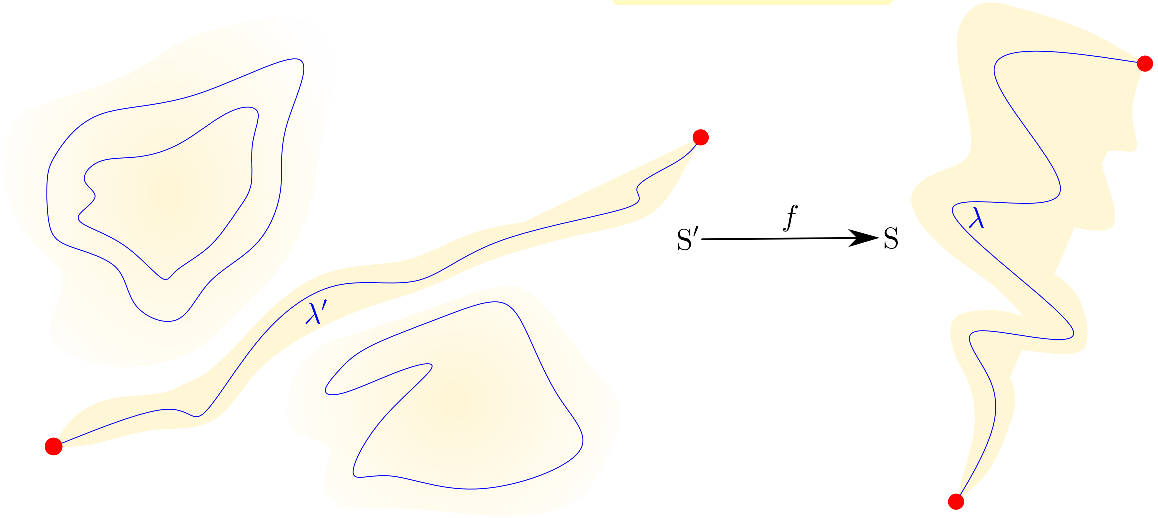
- The complexity of $S_{\mathsf{cut}} \coloneqq S \setminus \mathrm{int}(\lambda \times [-1, 1])$ is one less than the complexity of S.



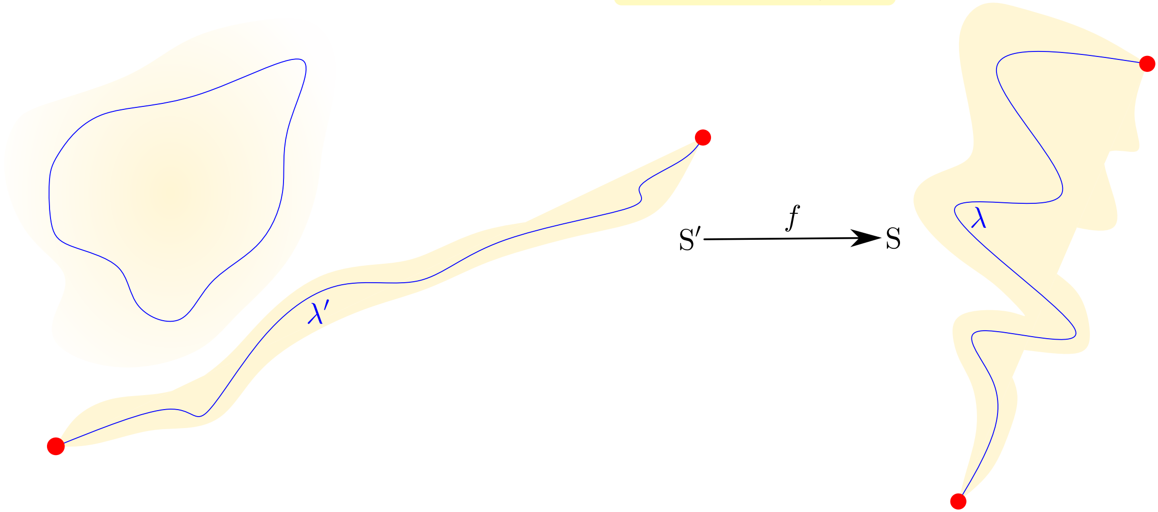
• Homotope f relative to $\partial S'$ so that $f \pitchfork \lambda$. Thus, $f^{-1}(\lambda)$ is a embedded one-dimensional compact submanifold of S' such that $f^{-1}(\lambda) \cap \partial S' = \partial f^{-1}(\lambda)$.



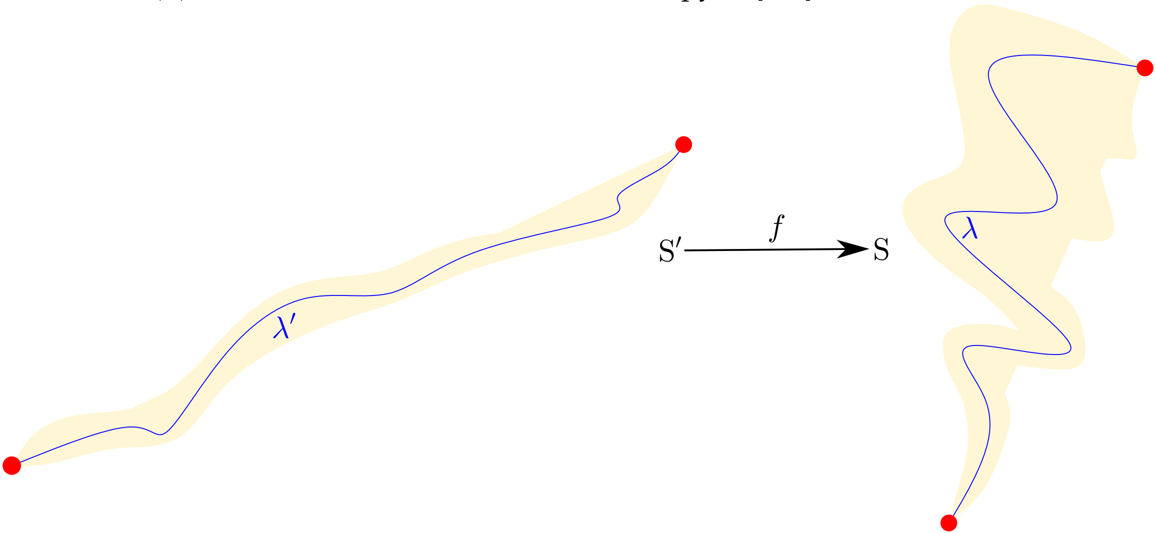
- Further, homotope f relative to $\partial S'$ to remove all circles one-by-one from $f^{-1}(\lambda)$.



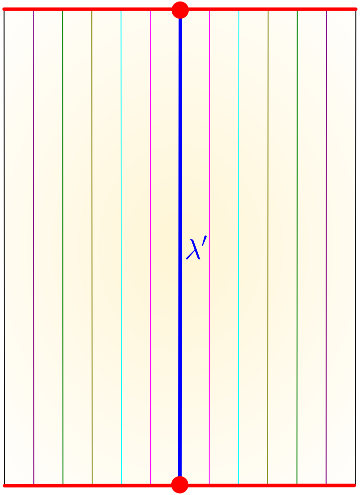
• Further, homotope f relative to $\partial \mathrm{S}'$ to remove all circles one-by-one from $f^{-1}(\lambda)$.



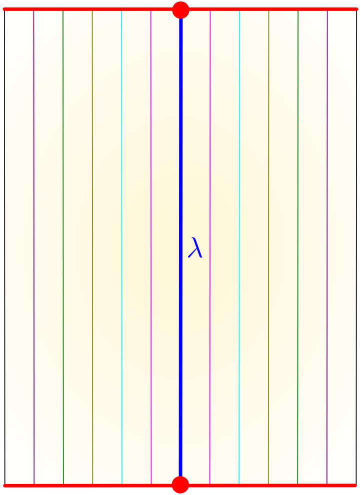
- Thus, $f^{-1}(\lambda)$ becomes λ', which is an embedded copy of $[0,1]$ in S', with $\lambda' \cap \partial \mathrm{S}' = \partial \lambda'$.



• Homotope f rel $\partial S'$ so that $f|\lambda' \to \lambda$ becomes a homeomorphism with the following properties.



$$\ni (x,t) \longmapsto \big(f(x),t\big) \in$$



$f^{-1}\big(\lambda \times [-1,1]\big) \equiv \lambda' \times [-1,1]$ $\qquad\qquad$ $\lambda \times [-1,1]$

- $S'_{\text{cut}} := S' \setminus \operatorname{int}(\lambda' \times [-1, 1])$ is connected.

  ☞ Otherwise, the (geometric) degree of f restricted to each component of S'_{cut} would be 0, and hence the (geometric) degree of f would also be 0.

- The inclusion $S'_{\text{cut}} \hookrightarrow S'$ is π_1-injective.

  ☞ By HNN-Seifert–van Kampen Theorem.

- $f|S'_{\text{cut}} \to S_{\text{cut}}$ is a map of degree one.

  ☞ Because $f|\partial S'_{\text{cut}} \to \partial S_{\text{cut}}$ is a homeomorphism.

- $f|S'_{\text{cut}} \to S_{\text{cut}}$ is a homotopy equivalence.

  ☞ Since maps of degree one are π_1-surjective.

- By the inductive hypothesis, the result follows.

Topological Rigidity of Compact Boundaryless Surfaces

Theorem. [Nielsen] If $f\colon S' \to S$ is a homotopy equivalence between two compact boundaryless surfaces, then f is homotopic to a homeomorphism.

Sketch of proof. Since f is a map of degree ± 1, there exists a disk $D \subset S$ such that $f|f^{-1}(D) \to D$ is a homeomorphism. Applying the boundary case to

$$f|S' \setminus \operatorname{int} f^{-1}(D) \to S \setminus \operatorname{int} D,$$

we are done.

Which Compact 3-Manifolds Have a Hierarchy?

Theorem. [Haken] Let M_1 be a Haken manifold with a non-empty boundary. Then there exists a sequence of triples

$$M_j, \quad F_j \subset M_j, \quad U(F_j) \subset M_j; \quad M_{j+1} = \overline{M_j \setminus U(F_j)},$$

where $j = 1, ..., n$, such that

- each M_j is connected, irreducible 3-manifold such that inclusion induced map $\pi_1(M_i) \to \pi_1(M_j)$ is injective if $j \leq i \leq n$.
- F_j is an incompressible surface with boundary in M_j, $U(F_j)$ is a tubular neighborhood of F_j in M_j.
- each component of M_{n+1} is a ball.

Theorem. [Waldhausen] Haken manifolds are topologically rigid.

What Happens in Dimension 4?

- **Simply connected, closed 4-manifolds:** Homotopy types are determined by intersection forms [Milnor], and homeomorphism types are determined by intersection forms and the Kirby-Siebenmann invariant [Freedman].

- **Compact, contractible 4-manifolds with boundary:** Every integral homology sphere appears as a boundary, and any homeomorphism between boundaries extends to a homeomorphism of the manifolds [Freedman].

- **Compact, aspherical 4-manifolds with boundary:** There are non-homeomorphic manifolds with homeomorphic boundaries and isomorphic π_1 [Davis-Hillman]. But, if π_1 is elementary amenable, then topological rigidity holds [Davis-Hillman].

Topological Rigidity in High Dimensions

- Any closed Riemannian manifold of $\dim \geq 5$ with non-positive sectional curvatures is topologically rigid [Farrell-Jones].
- Any $\mathbb{S}^n$ is topologically rigid [Smale-Stallings-Zeeman-Newman, $n \geq 5$] [Freedman, $n = 4$] [Perelman, $n = 3$] + [Hopf].
- Suppose that $k + d \neq 3$. Then $\mathbb{S}^k \times \mathbb{S}^d$ is topologically rigid if and only if both k and d are odd [Kreck-Lück].
- Let M^{4k+3} be a closed smooth manifold for $k \geq 1$ whose fundamental group has torsion. Then M is not topologically rigid [Chang-Weinberger].

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Thank You