The Goldman bracket characterizes homeomorphisms between non-compact surfaces

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Motivation

Let M and N be boundaryless n-manifolds. Suppose M is homotopy equivalent to N.

Question. Is M homeomorphic to N?

Question. Is every homotopy equivalence $M \to N$ homotopic to a homeomorphism?

Convention. Unless stated otherwise, all manifolds are connected, oriented, and without boundary; in most cases, they are non-compact.

Main Results

Let $\mathscr{S}$ be the set of all surfaces whose fundamental groups are not finitely generated.

Proposition. [Das] $\sharp\{\text{homotopy types in } \mathscr{S}\} = 1$ and $\sharp\{\text{homeomorphism types in } \mathscr{S}\} = 2^{\aleph_0}$.

Theorem A [Das] Let $f\colon \Sigma' \to \Sigma$ be a homotopy equivalence between any two elements of $\mathscr{S}$. If f is a *proper map*, then f is properly homotopic to a homeomorphism.

Application: Characterization of Homeomorphisms Between Infinite-Type Surfaces

Theorem B [Das-Gadgil-Nair] Let $f\colon \Sigma' \to \Sigma$ be a homotopy equivalence between any two elements of $\mathscr{S}$. Then,

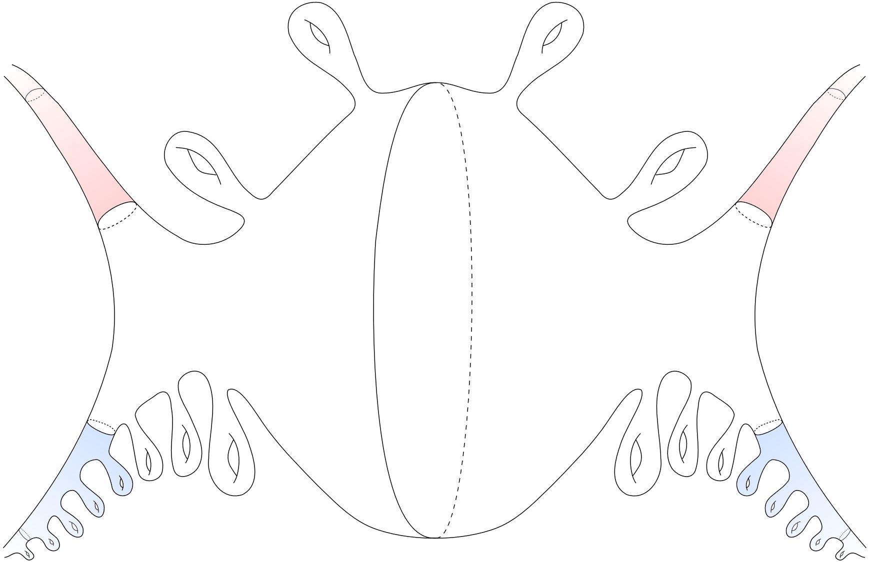
- f is homotopic to a homeomorphism if and only if f preserves the geometric intersection number.

- f is homotopic to an *orientation-preserving* homeomorphism if and only if f preserves the Goldman bracket.

Construction of Non-Compact Surfaces

Theorem. [Richards] Any non-compact surface is homeomorphic to a surface S obtained by attaching at most countably many handles to $\mathbb{S}^2 \setminus \mathscr{E}$, where $\mathscr{E}$ is a non-empty, closed, totally-disconnected subset of $\mathbb{S}^2$.

A surface with two planar ends and two non-planar ends:



Classification of Non-Compact Surfaces

Theorem. [Kerékjártó] Suppose S and S' are non-compact surfaces obtained by adding at most countably many handles to $\mathbb{S}^2 \setminus \mathscr{E}$ and $\mathbb{S}^2 \setminus \mathscr{E}'$, respectively, where $\mathscr{E}$ and , $\mathscr{E}'$ are two non-empty, closed, totally-disconnected subsets of $\mathbb{S}^2$.

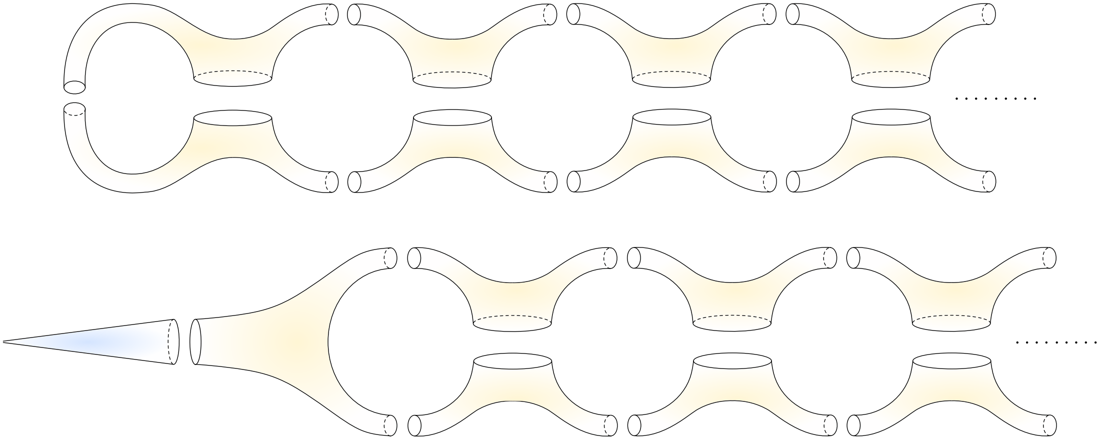
Let $\mathscr{E}_{\mathrm{np}} := \{p \in \mathscr{E} : p$ is accumulated by handles$\}$. Then S is homeomorphic to S' iff

- the number of handles of S is the same as the number of handles of S', and
- there is a homeomorphism $\varphi \colon \mathscr{E} \to \mathscr{E}'$ such that $\varphi(\mathscr{E}_{\mathrm{np}}) = \mathscr{E}'_{\mathrm{np}}$.

Corollary. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two non-empty closed subsets of the Cantor set. Then $\mathbb{S}^2 \setminus \mathscr{C}_1$ is homeomorphic to $\mathbb{S}^2 \setminus \mathscr{C}_2$ if and only if $\mathscr{C}_1$ is homeomorphic to $\mathscr{C}_2$.

Sketch of the Proof of Theorem A

1. Decompose Σ by a locally finite collection $\mathscr{C}$ of circles into copies of the pair of pants and copies of the punctured disk.



2. Modify the Whitney approximation and transversality homotopy theorems in the proper category, and then properly homotope f so that $f \pitchfork \mathscr{C}$.

- $f^{-1}(\mathscr{C})$ is a locally finite collection of circles.

  ☞ Since $\mathscr{C}$ is locally finite and f is proper.

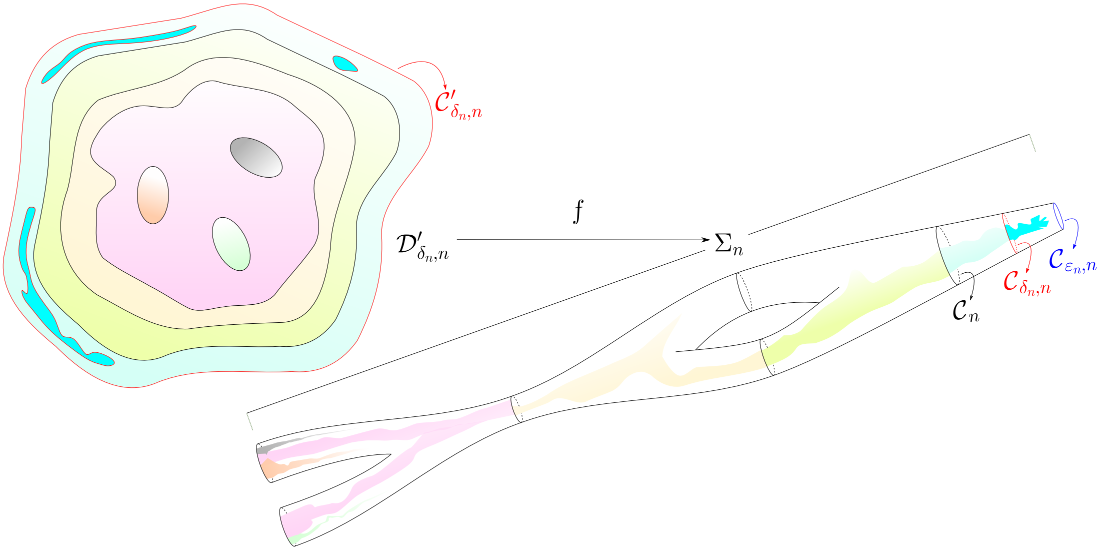
- $f^{-1}(\mathcal{C})$ can be empty for a component $\mathcal{C}$ of $\mathscr{C}$.

  ☞ A priori, it is not known whether $\deg(f) \neq 0$.

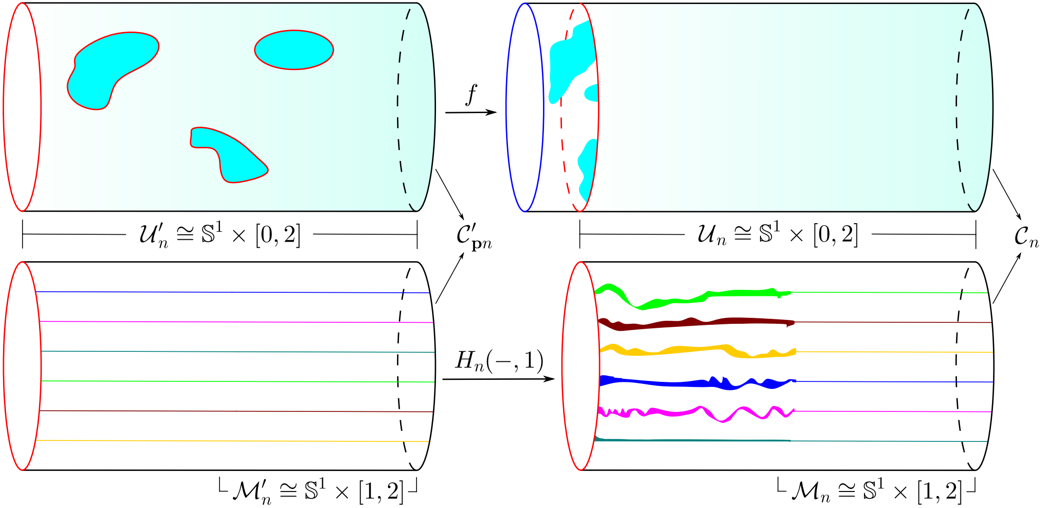
- There do not exist infinitely many components $\mathcal{C}'_1, \mathcal{C}'_2, \ldots$ of $f^{-1}(\mathscr{C})$ bounding disks $\mathcal{D}'_1, \mathcal{D}'_2, \ldots$ in Σ', with $\mathcal{C}'_n \subseteq \mathrm{int}(\mathcal{D}'_{n+1})$ for all n.

  ☞ Because $\Sigma' \neq \mathbb{R}^2$ and $f^{-1}(\mathscr{C})$ is locally finite.

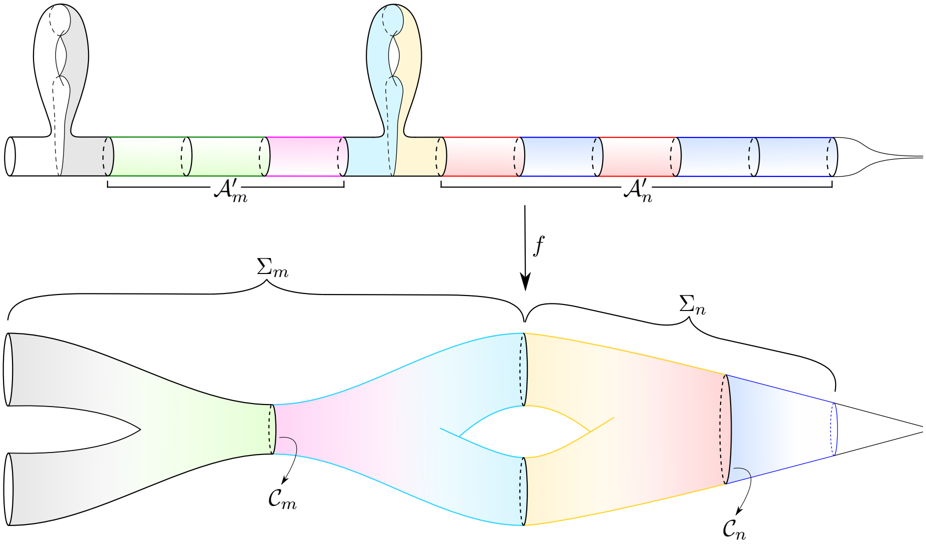
3. By considering all outermost disks simultaneously, properly homotope f to remove all trivial components from $f^{-1}(\mathscr{C})$.



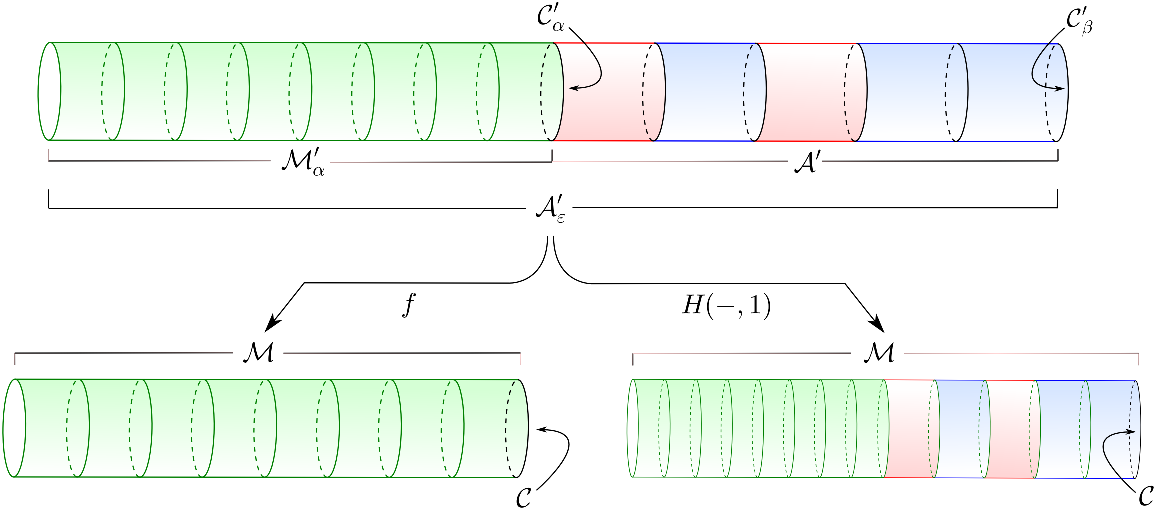
4. Now, f can be properly homotoped to map every small one-sided tubular neighbourhood of a component of $f^{-1}(\mathscr{C})$ onto a one-sided tubular neighbourhood of a component of $\mathscr{C}$ by a level-preserving homeomorphism.



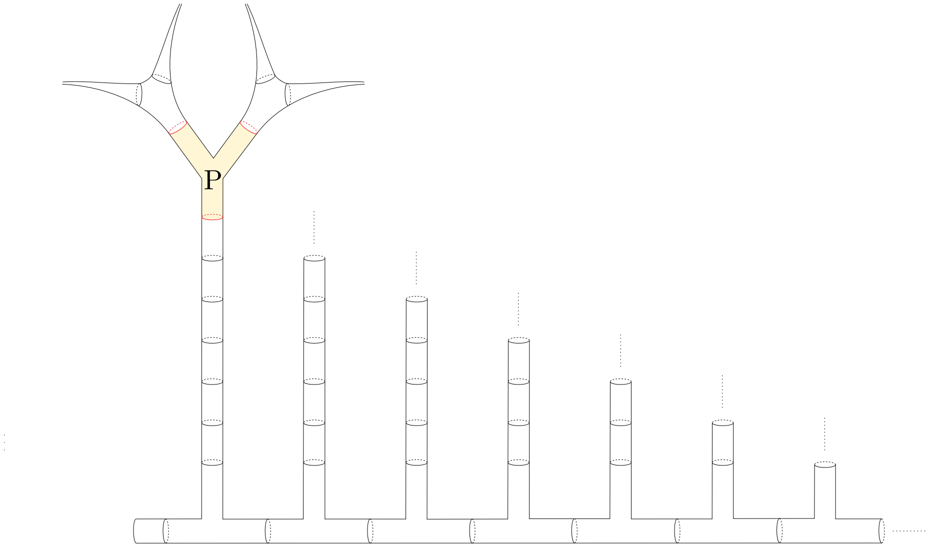
5. Properly homotope f so that each outermost annulus bounded by two components of $f^{-1}(\mathscr{C})$ can be mapped onto a component of $\mathscr{C}$.



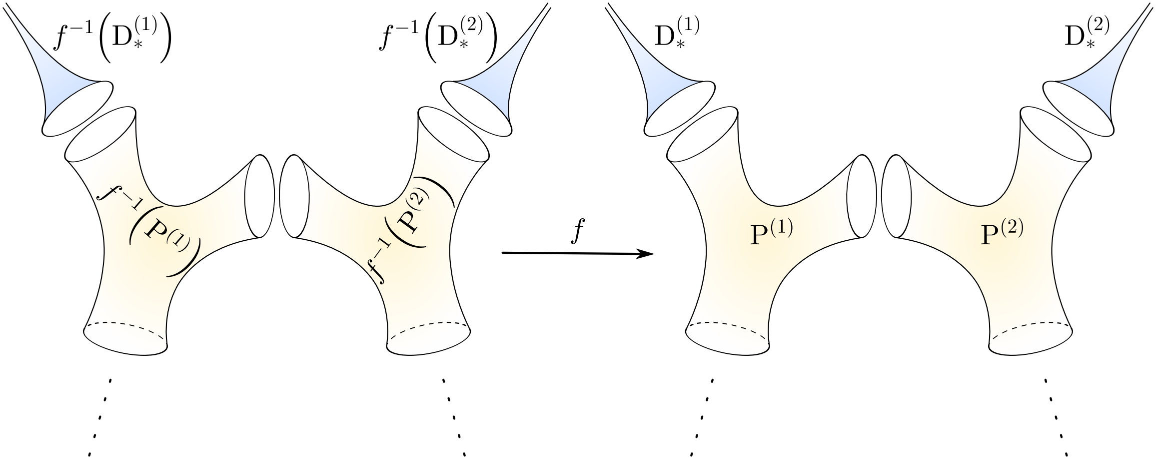
6. Now, properly homotope f to push each outermost annulus $\mathcal{A}'$ to an one-sided small tubular neighborhood $\mathcal{M}'_\alpha$ of a component $\mathcal{C}'_\alpha$ of $\partial\mathcal{A}'$ such that for each component $\mathcal{C}$ of $\mathscr{C}$, either $f^{-1}(\mathcal{C})$ is empty or $f|f^{-1}(\mathcal{C}) \to \mathcal{C}$ is a homeomorphism.



7. For an essential pair of pants P in Σ, after proper homotopies, we may assume that $f|f^{-1}(\partial \mathrm{P}) \to \partial \mathrm{P}$ and hence $f|f^{-1}(\mathrm{P}) \to \mathrm{P}$ are homeomorphisms, implying $\deg(f) = \pm 1$.



8. Hence, after a proper homotopy, $f|f^{-1}(\mathscr{C}) \to \mathscr{C}$ is a homeomorphism. Thus, for each complementary component S, $f|f^{-1}(\mathrm{S}) \to \mathrm{S}$ can be properly homotoped rel $\partial f^{-1}(\mathrm{S})$ to a homeomorphism $h_{\mathrm{S}}\colon f^{-1}(\mathrm{S}) \to \mathrm{S}$. Pasting all the h_{S}'s together completes the proof.



The Geometric Intersection Number

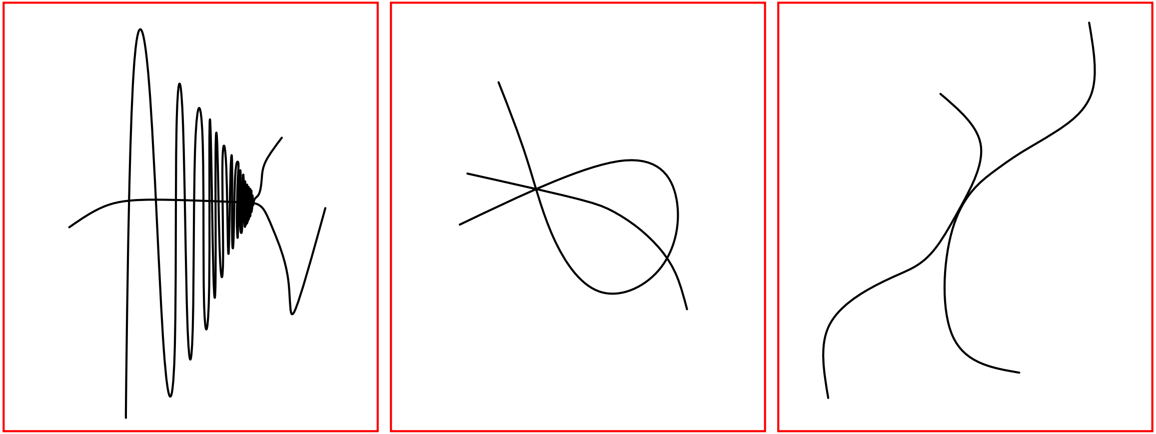
Let S be a surface. Denote the set (free) homotopy classes of maps $\mathbb{S}^1 \to S$ by $\widehat{\pi}(S)$.

Definition. The *geometric intersection number* $I_S \colon \widehat{\pi}(S) \times \widehat{\pi}(S) \to \mathbb{N} \cup \{0\}$ is defined as follows:

$$I_S(x, y) \coloneqq \min_{\varphi,\psi} \sharp(\varphi \cap \psi),$$

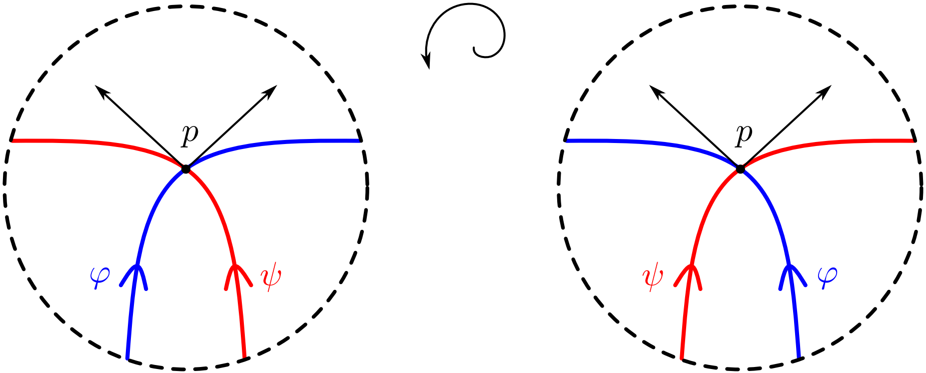
where φ and ψ are (immersed) representatives of x and y, respectively, such that φ and ψ *intersect transversally at double points*.

Intersections That Are Not Allowed



Oriented Intersection Numbers of Two Immersed Closed Curves

Definition. For two oriented immersed closed curves φ and ψ on an oriented surface S intersecting at p, define $\varepsilon(\varphi, \psi, p) = +1$ if φ intersects ψ at p as shown in the left-hand side figure; otherwise, let $\varepsilon(\varphi, \psi, p) = -1$.



The Goldman Lie algebra

Let S be an oriented surface. The *Goldman bracket* $[\cdot,\cdot]\colon \mathbb{Z}\widehat{\pi}(S) \times \mathbb{Z}\widehat{\pi}(S) \to \mathbb{Z}\widehat{\pi}(S)$ is a Lie algebra defined as follows:

For $x, y \in \widehat{\pi}(S)$, pick immersed representatives $\varphi \in x$ and $\psi \in y$ such that φ and ψ intersect tranversally at double points. Then

$$[x, y] := \sum_{p \in \varphi \cap \psi} \varepsilon(\varphi, \psi, p) \cdot [\varphi *_p \psi]\,,$$

where $[\varphi *_p \psi]$ is the free homotopy class of the based loop product.

Theorem. If $[x, y] = 0$ and x has a simple representative, then x and y have disjoint representatives.

Sketch of the Proof of Theorem B

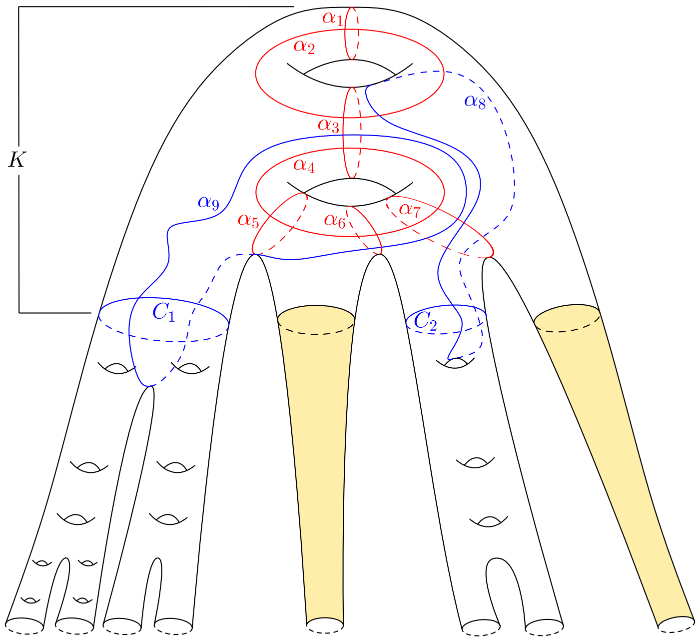
The If Direction of the Second Case

A Compact Exhaustion of an Infinite-Type Surface

Let Σ be an infinite-type surface.

Theorem. There exists a collection $K_1 \subset K_2 \subset \cdots \subset \Sigma$ of compact subsurfaces with boundary, called a *compact exhaustion of Σ*, such that K_1 is neither a disk nor an annulus, and $\bigcup_i K_i = \Sigma$. Moreover, for each i, the following hold:

- $K_i \subset \operatorname{int} K_{i+1}$,
- the closure of every component of $\Sigma \setminus K_i$ is non-compact, and
- every component of $\overline{\Sigma \setminus K_i}$ intersects K_i in a single component of ∂K_i.



(Extended) Filling System

Let K be an element of a compact exhaustion of Σ.

Theorem. There exists a finite collection $\{\alpha_1, \ldots, \alpha_n\}$, *called a filling system of K*, of simple closed curves on K such that if γ is a closed curve on K with $I_\Sigma(\gamma, \alpha_j) = 0$ for each j, then γ can be homotoped into $\Sigma \setminus K$.

For each component C of ∂K that does not bound an annulus, we pick a simple closed curve $\alpha \subset \Sigma$ such that $I_\Sigma(C, \alpha) \neq 0$. We extend $\{\alpha_1, \ldots, \alpha_n\}$ by adding these α's.

Pushing Loop Images Outside Subsurfaces

- Pick a compact exhaustion $\{K_i'\}$ of Σ'.

- For each element α of an extended filling system of K_i, select a closed curve $\alpha' \subset \Sigma'$ such that $f(\alpha')$ is homotopic to α. WLOG, assume each α' is contained in K_i'.

Theorem. Let γ' be a closed curve in Σ' that is homotopic to a closed curve in $\Sigma' \setminus K_i'$. Then $f(\gamma')$ is homotopic to a closed curve in $\Sigma \setminus K_i$.

Throwing an Intermediate Component into a Component of the Complement at the Fundamental Group Level

Theorem. Let V' be a component of $\overline{K_i' \setminus K_{i-1}'}$ with $\delta' = V' \cap K_{i-1}'$. Suppose V is a component of $\overline{\Sigma \setminus K_{i-1}}$ such that $f(\delta')$ is homotopic to a curve in V. Then, $\pi_1(f)$ sends $\pi_1(V') \lneq \pi_1(\Sigma')$ into $\pi_1(V) \leq \pi_1(\Sigma)$, where the base point is chosen from δ'.

An Inductive Process for Constructing a Proper Map Homotopic to f

Theorem. For each i, there is a map $g_i \colon K'_i \to \Sigma$ such that the following holds:

- $g_i|K'_{i-1} = g_{i-1}$.
- g_i is homotopic to $f|K'_i \to \Sigma$.
- $g_i(K'_i \setminus K'_{i-1}) \subset \overline{\Sigma \setminus K_{i-1}}$.

Theorem. The map $\lim g_i$ is a proper and homotopic to f. Therefore, f is homotopic to a homeomorphism by Theorem A.

Rigidity in the Proper Category

Definition. If a homotopy $\mathcal{H}: X \times [0,1] \to Y$ is a proper map, then we call $\mathcal{H}$ a *proper homotopy*.

Definition. We say that a proper map $f: X \to Y$ is a *proper homotopy equivalence* if there exists a proper map $g: Y \to X$ such that both $g \circ f$ and $f \circ g$ are properly homotopic to the identity maps.

Definition. An open n-manifold M is said to be *properly rigid* if, whenever N is another open n-manifold and $f: N \to M$ is a proper homotopy equivalence, then f is properly homotopic to a homeomorphism.

Properly Rigid Manifolds

- Any complete hyperbolic open manifold with finite-volume of dimension $\neq 3, 4, 5$ [Farrell-Jones].
- Any irreducible, end-irreducible, orientable, open 3-manifold M such that $\pi_1(M)$ is not isomorphic to the fundamental group of any compact surface [Brown-Tucker].
- Any $\mathbb{R}^n$ [Edwards-Perelman, $n = 3$] [Freedman, $n = 4$] [Siebenmann, $n \geq 5$] + [Epstein][a].
- Any finite-type non-compact surface [Nielsen].
- Any infinite-type surface [Das].

Proper Borel Conjecture. Every open aspherical manifold is properly rigid [Chang-Weinberger].

[a] Whitehead first provided an example of a contractible open 3-manifold that is not homeomorphic to $\mathbb{R}^3$.

Big Mapping Class Groups

Consider a surface S. Let $\mathrm{Homeo}(S)$ be the group of all self-homeomorphisms of S equipped with compact-open topology. Denote the path component of the identity in $\mathrm{Homeo}(S)$ by $\mathrm{Homeo}_0(S)$.

Definition. The extended mapping class group $\mathrm{MCG}^{\pm}(S)$ is the quotient space $\frac{\mathrm{Homeo}(S)}{\mathrm{Homeo}_0(S)}$.

Theorem. $\mathrm{MCG}^{\pm}(S)$ is a Polish group. Moreover, $\mathrm{MCG}^{\pm}(S)$ is discrete iff $S \notin \mathscr{S}$.

Theorem. If $S \in \mathscr{S}$, then $\mathrm{MCG}^{\pm}(S)$ is homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$.

Characterization of the image of the Dehn-Nielsen-Baer map

Corollary. If $\Sigma \in \mathscr{S}$, then each element of the image of the canonical injective group homomorphism

$$\mathrm{MCG}^{\pm}(\Sigma) \longrightarrow \mathrm{Out}\big(\pi_1(\Sigma)\big)$$

is induced by a homotopy equivalence $\Sigma \to \Sigma$ that preserves the geometric intersection number.

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Thank You