FREE LOOP SPACES AND TOPOLOGICAL COHOCHSCHILD HOMOLOGY I

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Goal: Describe a new approach to the homology of free loop spaces

$$H_*(\mathcal{L}X;k), \qquad \mathcal{L}X = \mathrm{Map}(S^1, X)$$

- via topological coHochschild homology.

Thm [Gromoll-Meyer 1969]: For M a simply connected closed smooth manifold, if for some field k the Betti numbers $\dim H_i(\mathcal{L}M;k)$ are unbounded then M has infinitely many distinct closed geodesics in any Riemannian metric.

One classic approach to $H_*(\mathcal{L}X)$ is through Hochschild homology.

Hochschild homology:

Let k be a commutative ring and A a k-algebra. Form a chain complex $C_\circ(A)$:

$$\begin{tikzcd}
\cdots \arrow[r] & A\otimes A\otimes A \arrow[r, "\partial"] & A\otimes A \arrow[r, "\partial"] & A \arrow[r] & 0
\end{tikzcd}$$

$\otimes = \otimes_k$

To define ∂:

$$\begin{tikzcd}
\cdots \arrow[r, shift left=4.5] \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] \arrow[r, shift right=4.5] & A\otimes A\otimes A \arrow[r, shift left=3, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=3, "d_2"'] & A\otimes A \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] & A \arrow[r] & 0
\end{tikzcd}$$

$$d_i : A^{\otimes q+1} \longrightarrow A^{\otimes q}$$

$$d_i(a_0\otimes a_1\otimes\cdots\otimes a_q) = \begin{cases} a_0\otimes\cdots a_i a_{i+1}\otimes\cdots\otimes a_q & 0\le i<q \\ a_q a_0\otimes\cdots\otimes a_{q-1} & i=q \end{cases}$$

$$d_i = \begin{cases} \mathrm{Id}^i \otimes \phi \otimes \mathrm{Id}^{q-i-1} & 0 \leq i < q \\ (\phi \otimes \mathrm{Id}^{q-1}) \circ \tau & i = q \end{cases}$$

τ cycles last element to the front.

$$\partial = \sum_i (-1)^i d_i, \qquad \partial^2 = 0$$

Example:

$$a \otimes b \otimes c \xmapsto{\partial} ab \otimes c - a \otimes bc + ca \otimes b$$

$$\xmapsto{\partial} abc - cab - (abc - bca) + cab - bca$$

$$= 0$$

$$\longrightarrow A \otimes A \otimes A \xrightarrow{\partial} A \otimes A \xrightarrow{\partial} A \longrightarrow 0$$

Defn: $HH_*(A) = H_*(C_*(A))$

Example: $HH_0(A) = A/\{ab-ba\} = A/[A,A]$

Two other perspectives on HH:

① If A is projective as a module over k:

$$HH_n(A) \cong \mathrm{Tor}_n^{A\otimes A^{op}}(A, A)$$

exercise in homological algebra.

② There is a simplicial perspective.

$$\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & A\otimes A\otimes A \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_2"'] & A\otimes A \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[r, shift left=1] \arrow[r, shift right=1] & A \arrow[l] \arrow[r] & 0 \\
 & 2 & 1 & 0 &
\end{tikzcd}$$

we can define a simplicial k-module $HH(A)_\bullet$ with q-simplices $HH(A)_q = A^{\otimes q+1}$

face maps are the maps d_i.

degeneracies:

$$s_i(a_0\otimes \cdots \otimes a_q) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \cdots \otimes a_q$$
$$0 \leq i \leq q$$

Dold Kan correspondence:

$$HH_*(A) = H_*(C_\bullet(A)) \cong \pi_*(|HH(A)_\bullet|)$$

[Goodwillie] [Burghelea-Fiedorowicz] 1985:

For X a pointed space

$$H_*(\mathcal{L}X) \cong HH_*(C_*(\Omega X))$$

singular chains

pointed loop space

This week: New approach to $H_*(\mathcal{L}X)$ via Hochschild invariants.

Idea: Look at Hochschild-type invariants for coalgebras.

Algebras	Coalgebras
Commutative ring k	Commutative ring k
Algebra A over k	Coalgebra C over k
product:	Coproduct
$\phi : A \otimes_k A \longrightarrow A$	$\Delta : C \longrightarrow C \otimes_k C$

Unit:

$\eta : k \longrightarrow A$

associativity:

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "id\otimes\phi"] \arrow[d, "\phi\otimes id"'] \arrow[dr, phantom, "\circlearrowleft"] & A\otimes A \arrow[d, "\phi"] \\
A\otimes A \arrow[r, "\phi"] & A
\end{tikzcd}$$

unitality:

$\phi\circ(id\otimes\eta) = id = \phi\circ(\eta\otimes id)$

Counit:

$\epsilon : C \longrightarrow k$

Coassociativity:

$$\begin{tikzcd}
C\otimes C\otimes C \arrow[dr, phantom, "\circlearrowright"] & C\otimes C \arrow[l, "id\otimes\Delta"'] \\
C\otimes C \arrow[u, "\Delta\otimes id"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

Counitality:

$(id\otimes\epsilon)\circ\Delta = id = (\varepsilon\otimes id)\circ\Delta.$

coHochschild homology [Doi 70's]

Let D be a coalgebra over a field k.

Define a cochain complex $C^*(D)$:

$$\leftarrow D\otimes D\otimes D \xleftarrow{\delta} D\otimes D \xleftarrow{\delta} D \leftarrow 0$$

where to define δ we exand

$$\begin{tikzcd} \cdots & D\otimes D\otimes D \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] & D\otimes D \arrow[l, shift left=2, "\delta_0"'] \arrow[l, "\delta_1"] \arrow[l, shift right=2, "\delta_2"] & D \arrow[l, shift left] \arrow[l, shift right] & 0 \arrow[l] \end{tikzcd}$$

$$\delta_i : D^{\otimes(q+1)} \longrightarrow D^{\otimes(q+2)}$$

$$\delta_i = \begin{cases} Id^i \otimes \Delta \otimes Id^{(q-i)} & 0 \leq i \leq q \\ \tau \circ (\Delta \otimes Id^q) & i = q+1 \end{cases}$$

τ cycles first entry to last.

$$\delta = \Sigma (-1)^i \delta_i$$

Def: $CoHH_q(D) = H^q(C^\bullet(D))$

Note:

① $CoHH_q(D) \cong CoTor^q_{D\otimes D^{op}}(D, D)$

② The cochain complex above comes from a cosimplicial k-module $CoHH(D)^\bullet$ with $CoHH(D)^q = D^{\otimes q+1}$

$$\begin{tikzcd}
\vdots \\
D\otimes D\otimes D \\
D\otimes D \arrow[u, shift left=4] \arrow[u, shift left=2, <-] \arrow[u] \arrow[u, shift right=2, <-] \arrow[u, shift right=4] \\
D \arrow[u, shift left=3] \arrow[u, <-] \arrow[u, shift right=3]
\end{tikzcd}$$

Cofaces are maps δ_i above

Codegeneracies

$$\sigma_i : D^{\otimes(q+2)} \longrightarrow D^{(q+1)}$$

$$\sigma_i = \mathrm{Id}^{(i+1)} \wedge \epsilon \wedge \mathrm{Id}^{(q-i)} \qquad 0 \leq i \leq q$$

Topological Hochschild homology [Bökstedt]

Idea: translate construction from algebra to topology.

Algebra	topology
Ring A	ring spectrum R mult: $\mu : R \wedge R \to R$ unit: $\eta : \mathbb{S} \to R$
$\otimes$	$\wedge$
ground ring $\mathbb{Z}$	sphere spectrum $\mathbb{S}$
$HH(A)$.	$THH(R)$.

For a ring spectrum R, form a simplicial spectrum

$$THH(R)_\bullet$$

$$THH(R)_q = R^{\wedge q+1}$$

face and degeneracies as before:

$$\begin{tikzcd} \vdots \\ R\wedge R\wedge R \arrow[d, shift left=4] \arrow[d] \arrow[d, shift right=4] \\ R\wedge R \arrow[u, shift left=2] \arrow[u, shift right=2] \arrow[d, shift left=2] \arrow[d, shift right=2] \\ R \arrow[u] \end{tikzcd}$$

$$d_q = \begin{cases} Id^{\wedge i} \wedge \mu \wedge Id^{\wedge q-i-1} & 0 \leq i < q \\ (\mu \wedge Id^{\wedge q-1}) \circ \tau & i = q \end{cases}$$

$$s_i = Id^{\wedge(i+1)} \wedge \eta \wedge Id^{\wedge(q-i)}$$

$$THH(R) = |THH(R)_\bullet|$$

Note. For a ring A we write

$THH(A)$ for $THH(HA)$ ↖ Eilenberg Maclane spectrum of A

Why?

Algebraic K-theory: Let A be a ring.

Let $K_n(A)$ denote algebraic K-groups of A.

Defn [Quillen]:

$$K_n(A) = \pi_n\left(BGL(A)^+\right), \quad n > 0$$

Problem: Alg. K-theory is very difficult to compute directly.

Trace methods: Approximate K theory by more computable things.

Dennis trace:

$$\begin{tikzcd}
 & HC^-_q(A) \arrow[dr] & \\
K_q(A) \arrow[ur] \arrow[rr] & & HH_q(A)
\end{tikzcd}$$

Goodwillie: In good situations $HC^-_q(A)$ tells us about the rank of algebraic K-theory.

Would like a topological analogue of these invariants, that can capture torsion information.

$$K_q(A) \longrightarrow \pi_q THH(A)$$

topological Hochschild homology

- from THH can define topological cyclic homology, TC, which is often quite close to algebraic K-theory.

$$\begin{tikzcd}
\text{Hochschild homology} \arrow[r, squiggly] \arrow[d, squiggly] & \text{Topological Hochschild homology [Bökstedt]} \arrow[d, squiggly] \\
\text{CoHochschild homology [Doi]} \arrow[r, squiggly] & \text{topological coHochschild homology coTHH [Hess-Shipley 2018].}
\end{tikzcd}$$

Topological CoHochschild homology

Algebra	Topology
Coalgebra D	Coalgebra in spectra
$\otimes$	$\wedge$
$\mathrm{CoHH}^{\bullet}(D)$	$\mathrm{CoTHH}^{\bullet}(C)$

$\Delta : C \longrightarrow C \wedge C$

$\epsilon : C \longrightarrow \mathbb{S}$

$\mathrm{CoTHH}^{\bullet}(C)$

$$\begin{tikzcd}
C \wedge C \wedge C \arrow[d, shift left=1] \arrow[d, shift right=1] \\
C \wedge C \arrow[u, shift left=3] \arrow[u] \arrow[u, shift right=3] \arrow[d] \\
C \arrow[u, shift left=2] \arrow[u, shift right=2]
\end{tikzcd}$$

$$\delta_i = \begin{cases} \mathrm{Id}^i \otimes \Delta \otimes \mathrm{Id}^{(q-i)} & 0 \leq i \leq q \\ \tau \circ (\Delta \otimes \mathrm{Id}^q) & i = q+1 \end{cases}$$

$$\sigma_i = \mathrm{Id}^{(i+1)} \wedge \epsilon \wedge \mathrm{Id}^{(q-i)} \qquad 0 \leq i \leq q$$

$$\mathrm{CoTHH}(C) = \mathrm{Tot}(\mathrm{CoTHH}^\bullet(C))$$