Free Loop Spaces and Topological CoHochschild Homology II

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Goal: Describe a new approach to the homology of free loop spaces

$$H_*(\mathcal{L}X;k), \quad \mathcal{L}X = \mathrm{Map}(S^1, X)$$

$\rightsquigarrow$ via topological coHochschild homology.

$$\begin{tikzcd}
\text{Hochschild homology for algebras} \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & \text{Topological Hochschild homology [Bökstedt]} \arrow[d, rightsquigarrow] \\
\text{coHochschild homology for coalgebras [Doi 70's]} \arrow[r, rightsquigarrow] & \text{topological coHochschild homology [Hess-Shipley 2018]}
\end{tikzcd}$$

last time:

For C a coalgebra in spectra, we defined

$$\mathrm{coTHH}(C)$$

Example:

For a space X, the suspension spectrum $\Sigma^\infty_+ X$ is a coalgebra.

$$\Delta : X \longrightarrow X \times X$$

yields comultiplication:

$$\Sigma^\infty_+ X \longrightarrow \Sigma^\infty_+ X \wedge \Sigma^\infty_+ X$$

Thm [Malkiewich, Kuhn, Hess-Shipley]:

For X simply connected

$$\mathrm{coTHH}(\Sigma^\infty_+ X) \simeq \Sigma^\infty_+ \mathcal{L} X$$

Further:

Thm [Bökstedt - Waldhausen]:

$$\mathrm{THH}(\Sigma^\infty_+ (\Omega X)) \simeq \Sigma^\infty_+ \mathcal{L} X$$

So, for X a simply connected space

$$\mathrm{coTHH}(\Sigma^\infty_+ X) \simeq \mathrm{THH}(\Sigma^\infty_+ (\Omega X))$$

$$A(X) \simeq K(\Sigma^\infty_+ \Omega X) \longrightarrow THH(\Sigma^\infty_+ \Omega X) \underset{\text{s.c.}}{\simeq} coTHH(\Sigma^\infty_+ X)$$

Waldhausen's algebraic K-theory of spaces.

Q: How do we compute the homology of $coTHH(\Sigma^\infty_+ X)$?

If we could do this, we would have a method of computing $H_*(\mathcal{L}X; k)$.

Q: How does one compute the homology of $THH(R)$?

Recall: $THH(R)$ is the geometric realization of a simplicial spectrum $THH(R)_\bullet$.

$$\begin{array}{c} \vdots \\ R \wedge R \wedge R \\ \downarrow\uparrow\downarrow\uparrow\downarrow \\ R \wedge R \\ \downarrow\uparrow\downarrow \\ R \end{array}$$

Fact: [EKMM] under mild hypotheses, for a simplicial spectrum $X_\bullet$ and a field k there is a spectral sequence:

$$E^2_{p,q} = H_p(H_q(X_\bullet; k)) \Longrightarrow H_*(|X_\bullet|; k)$$

Idea: Use skeletal filtration on $X_\bullet$ to get an exact couple + hence a spectral sequence.

Consider $THH(R)_\bullet$

$$E^2_{p,q} = H_p(\underbrace{H_q(THH(R)_\bullet; k)}_{\text{what is this?}}) \Rightarrow H_*(THH(R); k)$$

$$\vdots$$

$$\begin{array}{ccc} H_*(R \wedge R \wedge R; k) & \text{Kunneth formula} & H_*(R;k) \otimes_k H_*(R;k) \otimes_k H_*(R;k) \\ \downarrow \downarrow \downarrow & \rightsquigarrow & \downarrow \downarrow \downarrow \\ H_*(R \wedge R; k) & \text{since field coeffs} & H_*(R;k) \otimes_k H_*(R;k) \\ \downarrow \downarrow & & \downarrow \downarrow \\ H_*(R; k) & & H_*(R; k) \end{array}$$

- need to take homology of the chain complex associated to this simplicial k-module to get

$$H_p(H_q(THH(R)_\bullet; k))$$

Claim: The boundary in this chain complex is the Hochschild differential.

Bökstedt spectral sequence.

$$E^2_{*,*} = \underbrace{HH_*(H_*(R;k))}_{\text{algebra}} \Longrightarrow H_*(THH(R);k)$$

Q: Do we have an analogous spectral sequence for coTHH?

Approach:

- Consider the Bousfield-Kan spectral sequence arising from the cosimplicial spectrum $coTHH^\bullet(C)$.
- Try to identify E_2-term.
- Determine conditions under which this spectral sequence converges.

Thm [Bohmann-G-Høgenhaven-Shipley-Ziegenhagen]
Let k be a field, C a coalgebra spectrum. There is a "coBökstedt" spectral sequence for calculation $H_*(\mathrm{coTHH}(C);k)$ with E_2-term

$$E_2^{*,*} = \mathrm{CoHH}_*(H_*(C;k))$$

Algebraic theory

Note: For C a coalgebra in spectra, the homology of C with field coeffs is a coalgebra.

$$C \longrightarrow C \wedge C$$

$$\Delta: H_*(C;k) \longrightarrow H_*(C\wedge C;k) \cong H_*(C;k)\otimes_k H_*(C;k)$$

Look at case $C = \Sigma^\infty_+ X$

Prop [BGHSZ]: Let X be a simply connected space. If for each s there is an r such that

$$E_r^{s,s+i} = E_\infty^{s,s+i} \quad \text{then}$$

$$E_2^{**} = \mathrm{CoHH}_*(H_*(X;k)) \Rightarrow H_*(\mathcal{L}X;k)$$

Goal: Produce new calculations of

$H_*(\mathcal{L}X;k)$.

Goal: Understand the algebraic structure in the CoBökstedt spectral sequence.

Starting point: classical Bökstedt spectral sequence.

Prop [Angeltveit-Rognes]: If A is commutative and $HH_*(A)$ is flat as an A-module then $HH_*(A)$ is an A-Hopf algebra.

Recall For a commutative ring A, an A-Hopf algebra H is an A-module with:

- $\phi: H \otimes_A H \longrightarrow H$ product ← associative + unital
- $\eta: A \longrightarrow H$ unit
- $\psi: H \longrightarrow H \otimes_A H$ Coproduct ← Coassociative + counital
- $\epsilon: H \longrightarrow A$ counit
- $\chi: H \longrightarrow H$ antipode

satisfying various compatibilities . . .

e.g.

$$\begin{tikzcd}
H \otimes_A H \arrow[r, "\Phi"] \arrow[d, "\psi \otimes \psi"'] & H \arrow[r, "\psi"] & H \otimes_A H \\
H \otimes_A H \otimes_A H \otimes_A H \arrow[rr, "id \otimes \gamma \otimes id"] & & H \otimes_A H \otimes_A H \otimes_A H \arrow[u, "\phi \otimes \phi"']
\end{tikzcd}$$

Q: Why is $HH_*(A)$ an A-Hopf algebra?

Idea: For a commutative ring A, identify $HH(A)_\bullet$ as a simplicial tensor:

$$HH(A)_\bullet = A \otimes S^1_\bullet$$

For X a finite set

$$A \otimes X = \bigotimes_{x \in X} A$$

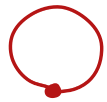
There is a simplicial model of $S^1_\bullet$ where S^1_q has $q+1$ elements.

The face and degeneracy maps in $S^1_\bullet$ induce face

+ degeneracy maps on $A \otimes S^1_\bullet$

<u>Claim:</u> $A \otimes S^1_\bullet \cong HH(A)_\bullet$

Simplicial maps on $S^1_\bullet$ induce maps on Hochschild homology.



<u>Simplicial map:</u> $\rightsquigarrow$ <u>Map on $HH_\bullet(A)$</u>

$\nabla : S^1_\bullet \vee S^1_\bullet \longrightarrow S^1_\bullet \rightsquigarrow \phi : HH_\bullet(A) \otimes_A HH_\bullet(A) \to HH_\bullet(A)$

fold

$\eta : \ast \longrightarrow S^1_\bullet \rightsquigarrow \eta : A \longrightarrow HH_\bullet(A)$

$\epsilon : S^1_\bullet \longrightarrow \ast \rightsquigarrow \epsilon : HH_\bullet(A) \longrightarrow A$

$\psi : dS^1_\bullet \longrightarrow S^1_\bullet \vee S^1_\bullet \rightsquigarrow \psi : HH_\bullet(A) \to HH_\bullet(A) \otimes_A HH_\bullet(A)$

pinch

Need a "double" simplicial model of S^1

$dS^1_\bullet$

$\chi : dS^1_\bullet \to dS^1_\bullet \rightsquigarrow \chi : HH_\bullet(A) \longrightarrow HH_\bullet(A)$

flip

These maps give HH the structure of a Hopf algebra over A.

For A commutative ring spectrum $THH(A) \simeq A \otimes S^1$.

[McClure-Schwänzl-Vogt] [Angeltveit-Rognes] For A commutative, $THH(A)$ is an A-Hopf algebra in the homotopy category.

Next up: Algebraic structure descend to Bökstedt spectral sequence.

Compute homology of free loop spaces.