FREE LOOP SPACES AND TOPOLOGICAL COHOCHSCHILD HOMOLOGY III

Joint with: Anna Marie Bohmann and Brooke Shipley

Goal: New approach to

$$H_*(\mathcal{L}X;k)$$

$\rightsquigarrow$ via topological coHochschild homology

For C a coalgebra in spectra, defined $\mathrm{coTHH}(C)$

Key example: For X a space, $\Sigma^\infty_+ X$ is a coalgebra in spectra.

For X simply connected

$$\mathrm{coTHH}(\Sigma^\infty_+ X) \simeq \Sigma^\infty_+ \mathcal{L}X$$

Q: How do we compute homology of coTHH?

<u>Last time</u>: Hopf algebra structures in Hochschild homology.

<u>Prop</u>: [Angeltveit-Rognes]: If A is commutative and $HH_*(A)$ is flat as an A-module then $HH_*(A)$ is an A-Hopf algebra.

<u>IDea</u>: $HH(A)_\bullet = A \otimes S^1_\bullet$

$\rightsquigarrow$ use maps on $S^1_\bullet$ to induce maps on $HH(A)$.

Further for A commutative ring spectrum

$$THH(A) \simeq A \otimes S^1_\bullet$$

and $THH(A)$ is an A-Hopf algebra in the homotopy category.

<u>Today</u>: Algebraic structure in Bökstedt | coBökstedt spectral sequences $\to$ the homology of free loop spaces.

Then this algebraic structure descends to Bökstedt spectral sequence:

$$E^2_{*,*} = HH_*(H_*(R;k)) \Longrightarrow H_*(THH(R);k)$$

Thm [Angeltveit - Rognes]: Let R be a commutative ring spectrum. In the Bökstedt spectral sequence for $H_*(THH(R);k)$ if each term $E^r_{*,*}$ $r \geq 2$ is flat over $H_*(R;k)$ then it is a spectral sequence of Hopf algebras over $H_*(R;k)$.

i.e. has a product and a coproduct satisfying

Leibniz rule: $\quad d(xy) = d(x)y \pm x d(y)$

Coleibniz rule: $\quad \psi \circ d = (d \otimes 1 \pm 1 \otimes d) \circ \psi$

Hope: we would like similar structure in coBökstedt spectral sequence.

Idea: $\mathrm{CoHH}^\bullet(D)$ and $\mathrm{CoTHH}^\bullet(D)$ can be viewed as cosimplicial cotensors of D with $S^1_\bullet$, and maps of circle induce maps on these cosimplicial objects.

Two immediate issues:

- After totalization $\mathrm{CoTHH}(D)$ is not the cotensor with S^1

  Issue: $\mathrm{CoTHH}(D)$ is not naturally a coalgebra.

  $\rightsquigarrow$ arises from fact that totalization does not commute with smash product.

Resolution:

- for general coalgebras, work cosimplicially before totalization.

- For $C = \Sigma^\infty_+ X$, X simply connected, we do have a coalgebra structure on $\mathrm{CoTHH}(\Sigma^\infty_+ X)$.

- Algebraic structure on coBökstedt spectral sequence should be over $\underbrace{H_*(C;k)}_{\text{Coalgebra}}$

Resolution: define the notion of a $\square$-Hopf algebra over a coalgebra.

Q: How do we define a Hopf-like structure over a coalgebra?

Def: For C a coalgebra over a field k, a right C-comodule M is a k-module w/ linear coaction map

$$\rho_M : M \longrightarrow M \otimes C$$

that ~~coassociative~~ coassociative + counital.

Def: For M a right C-comodule and N a left C-comodule the cotensor product $M \square_C N$ is the equalizer:

$$M \square_C N \to M \otimes N \overset{\rho_M \otimes id}{\underset{id \otimes \rho_N}{\rightrightarrows}} M \otimes C \otimes N$$

Def: A right C-comodule is coflat if $M \square_C -$ is an exact functor.

Def: Let C be a cocommutative coalgebra over a field k. A $\square_C$-Hopf algebra is a C bicomodule H together with maps of bicomodules:

- $\Delta : H \longrightarrow H \square_C H$ coproduct ← coassociative + counital
- $\epsilon : H \longrightarrow C$ counit
- $\mu : H \square_C H \longrightarrow H$ product ← associative + unital
- $\eta : C \longrightarrow H$ unit
- $\chi : H \longrightarrow H$ antipode

satisfying various compatibilities.

Thm [Bohmann-G-Shipley]: For X a simply connected space and k a field if $H_*(\Sigma X;k)$ is coflat as a comodule over $H_*(X;k)$ the $H_*(\Sigma X;k)$ is a

$$\square_{H_*(X;k)}\text{-Hopf algebra.}$$

Prop [BGS]: Let D be a cocommutative coalgebra over a field k. If $coHH_*(D)$ is coflat over D then $coHH_*(D)$ is a $\square_D$-bialgebra

Recall: The coBökstedt spectral sequence computing $H_*(coTHH(C);k)$ had E_2-term

$$E_2^{*,*} = CoHH_*(H_*(C;k))$$

Thm [BGS]: Let C be a connected cocommutative coalgebra in spectra. If for each $r \geq 2$ $E_r^{*,*}$ is coflat over $H_*(C;k)$ then the coBökstedt spectral sequence is a spectral sequence of $\square_{H_*(C;k)}$-Hopf algebras.

How does this help?

Prop [BGS]: Under coflatness conditions

$E_2^{**} = \mathrm{CoHH}(H_*(C,k))$ is $\square_{H_*(C,k)}$-bialgebra and the shortest nonzero differential in lowest total degree maps from a $\square_{H_*(C,k)}$-algebra indecomposable to a $\square_{H_*(C;k)}$-coalgebra primitive.

Goal: Compute $H_*(\mathcal{L}X,k)$.

Give straightforward computations of

$H_*(\mathcal{L}X,k)$ for $X = \mathbb{C}P^\infty$, $BU(n)$, $BSU(n)$, $BSp(n)$...

[BGHSZ]

These are spaces w/ polynomial cohomology.

Harder question: homology of free loop spaces for simply connected spaces with exterior cohomology

[Kuribayashi - Yamaguchi, 1997]:

For X simply connected with

$$H^*(X;\mathbb{Z}/p) = \Lambda_{\mathbb{Z}/p}(x_{i_1}, x_{i_2}) \qquad |x_i| = i \text{ odd}$$

where $i_1 \le i_2 \le 2i_1 - 2$ and $p \geq 3$ they computed $H_*(\mathcal{L}X;\mathbb{Z}/p)$.

Thm [BGS]: Let k be a field of characteristic p and X a simply connected space whose cohomology is

$$H^*(X;k) = \Lambda_k(x_{i_1}, \dots x_{i_n}) \qquad |x_i| = i \text{ odd}$$

$i_j \le i_{j+1}$.

When $\dfrac{i_n + \sum_{j=1}^{n} i_j}{i_1 - 1} \le p$,

$$H_*(\Omega X; k) \cong \Lambda_k(y_{i_1}, \dots y_{i_n}) \otimes k[w_{i_1}, \dots w_{i_n}]$$

as a graded module, where

$$|y_i| = i, \quad |w_i| = i-1.$$

- identify D-indecomposables + D-primitives + use strong alg. structure in Eilenberg–Moore s.s. to make these computations possible.